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STEPHEN GUSTAFSON, STEFAN LE COZ, AND TAI-PENG TSAI

Abstract. We study the stability of the cnoidal, dnoidal and snoidal elliptic functions as spatially-periodic standing wave solutions of the 1D cubic nonlinear Schrödinger equations. First, we give global variational characterizations of each of these periodic waves, which in particular provide alternate proofs of their orbital stability with respect to same-period perturbations, restricted to certain subspaces. Second, we prove the spectral stability of the cnoidal waves (in a certain parameter range) and snoidal waves against same-period perturbations, thus providing an alternate proof of this (known) fact, which does not rely on complete integrability. Third, we give a rigorous version of a formal asymptotic calculation of Rowlands to establish the instability of a class of real-valued periodic waves in 1D, which includes the cnoidal waves of the 1D cubic focusing nonlinear Schrödinger equation, against perturbations with period a large multiple of their fundamental period. Finally, we develop a numerical method to compute the minimizers of the energy with fixed mass and momentum constraints. Numerical experiments support and complete our analytical results.

Contents

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1. Introduction

We consider the cubic nonlinear Schrödinger equation

$$i\psi_t + \psi_{xx} + b|\psi|^2\psi = 0, \qquad \psi(0,x) = \psi_0(x) \tag{1.1}$$

in one space dimension, where $\psi : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ and $b \in \mathbb{R}\setminus\{0\}$. Equation (1.1) has well-known applications in optics, quantum mechanics, and water waves, and serves as a model for nonlinear dispersive wave phenomena more generally [11, 31]. It is said to be *focusing* if $b > 0$ and *defocusing* if $b < 0$. Note that (1.1) is invariant under

- spatial translation: $\psi(t,x) \mapsto \psi(t,x+a)$ for $a \in \mathbb{R}$
- phase multiplication: $\psi(t,x) \mapsto e^{i\alpha}\psi(t,x)$ for $\alpha \in \mathbb{R}$.

We are particularly interested in the spatially periodic setting

$$\psi(t,\cdot) \in H^1_{\text{loc}} \cap P_T, \qquad P_T = \{f \in L^2_{\text{loc}}(\mathbb{R}) : \, f(x+T) = f(x)\ \forall x \in \mathbb{R}\}.$$

The Cauchy problem (1.1) is globally well-posed in $H^1_{\text{loc}} \cap P_T$ [7]. We refer to [6] for a detailled analysis of nonlinear Schrödinger equations with periodic boundary conditions. Solutions to (1.1) conserve mass $\mathcal{M}$, energy $\mathcal{E}$, and momentum $\mathcal{P}$:

$$\mathcal{M}(\psi) = \frac{1}{2}\int_0^T |\psi|^2 dx, \quad \mathcal{P}(\psi) = \frac{1}{2}\mathcal{I}m\int_0^T \psi\bar{\psi}_x dx,$$
$$\mathcal{E}(\psi) = \frac{1}{2}\int_0^T |\psi_x|^2 dx - \frac{b}{4}\int_0^T |\psi|^4 dx.$$

By virtue of its complete integrability, (1.1) enjoys infinitely many higher (in terms of the number of derivatives involved) conservation laws [27], but we do not use them here, in order to remain in the energy space H^1_{loc}, and with the aim of avoiding techniques which rely on integrability.

The simplest non-trivial solutions of (1.1) are the *standing waves*, which have the form

$$\psi(t,x) = e^{-iat}u(x), \qquad a \in \mathbb{R}$$

and so the profile function $u(x)$ must satisfy the ordinary differential equation

$$u_{xx} + b|u|^2u + au = 0. \tag{1.2}$$

We are interested here in those standing waves $e^{-iat}u(x)$ whose profiles $u(x)$ are spatially periodic – which we refer to as *periodic waves*. One can refer to the book [3] for an overview of the role and properties of periodic waves in nonlinear dispersive PDEs.

Non-constant, real-valued, periodic solutions of (1.2) are well-known to be given by the Jacobi elliptic functions: dnoidal (dn), cnoidal (cn) (for $b > 0$) and snoidal (sn) (for $b < 0$) – see Section 2 for details. To make the link with Schrödinger equations set on the whole real line, one can see a periodic wave as a special case of infinite train solitons [25, 26]. Another context in which periodic waves appear is when considering the nonlinear Schrödinger equation on a Dumbbell graph [28]. Our interest here is in the stability of these periodic waves against periodic perturbations whose period is a multiple of that of the periodic wave.

Some recent progress has been made on this stability question. By Grillakis-Shatah-Strauss [18, 19] type methods, orbital stability against energy (H^1_{loc})-norm perturbations of the same period is known for dnoidal waves [2], and for snoidal waves [13] under the additional constraint that perturbations are anti-symmetric with respect to the half-period. In [13], cnoidal waves are shown to be orbitally stable with respect to half-anti-periodic perturbations, provided some condition is satisfied. This condition is verified analytically for small amplitude cnoidal waves and numerically for larger amplitude. Remark here that the results in [13] are obtained in a broader setting, as they are also considering *non-trivially complex-valued* periodic waves. Integrable systems methods introduced in [5] and developed in [15] – in particular conservation of a higher-order functional – are used to obtain the orbital stability of the snoidal waves against H^2_{loc} perturbations of period *any* multiple of that of sn.

Our goal in this paper is to further investigate the properties of periodic waves. We follow three lines of exploration. First, we give *global* variational characterization of the waves in the class of periodic or half-anti-periodic functions. As a corollary, we obtain orbital stability results for periodic waves. Second, we prove the spectral stability of cnoidal, dnoidal and snoidal waves within the class of functions whose period is the fundamental period of the wave. Third, we prove that cnoidal waves are linearly unstable if perturbations are periodic for a sufficiently large multiple of the fundamental period of the cnoidal wave.

Our first main results concern global variational characterizations of the elliptic function periodic waves as constrained-mass energy minimizers among (certain subspaces of) periodic functions, stated as a series of Propositions in Section 3. In particular, the following characterization of the cnoidal functions seems new. Roughly stated (see Proposition 3.4 for a precise statement):

Theorem 1.1. *Let $b > 0$. The unique (up to spatial translation and phase multiplication) global minimizer of the energy, with fixed mass, among half-anti-periodic functions is a (appropriately rescaled) cnoidal function.*

Due to the periodic setting, existence of a minimizer for the problems that we are considering is easily obtained. The difficulty lies within the identification of this minimizer: is it a plane wave, a (rescaled) Jacobi elliptic function, or something else? To answer this question, we first need to be able to decide whether the minimizer can be considered real-valued after a phase change. This is far from obvious in the half-anti-periodic setting of Theorem 1.1, where we use a Fourier coefficients rearrangement argument (Lemma 3.5) to obtain this information. To identify the minimizers, we use a combination of spectral and Sturm-Liouville arguments.

As a corollary of our global variational characterizations, we obtain orbital stability results for the periodic waves. In particular, Theorem 1.1 implies the orbital stability of all cnoidal waves in the space of half-anti-periodic functions. Such orbital stability results for periodic waves were already obtained in [2, 13] as consequences of *local* constrained minimization properties. Our global variational characterizations provide alternate proofs of these results – see Corollary 3.9 and Corollary 4.7. The orbital stability of cnoidal waves was proved only for small amplitude in [13], and so we extend this result to all amplitude. Remark however once more that we are in this paper considering only real-valued periodic wave profiles, as opposed to [13] in which truly complex valued periodic waves were investigated.

Our second main result proves the linear (more precisely, *spectral*) stability of the snoidal and cnoidal (with some restriction on the parameter range in the latter case) waves against same-period perturbations, but *without* the restriction of half-period antisymmetry:

Theorem 1.2. *Snoidal waves and cnoidal waves (for a range of parameters) with fundamental period T are spectrally stable against T-periodic perturbations.*

See Theorem 4.1 for a more precise statement. For sn, this is already a consequence of [5, 15], whereas for cn the result was obtained in [21]. The works [5, 15] and [21] both exploit the integrable structure, so our result could be considered an alternate proof which does not uses integrability, but instead relies mainly on an invariant subspace decomposition and an elementary Krein-signature-type argument. See also the recent work [16] for related arguments.

The proof of Theorem 1.2 goes as follows. The linearized operator around a periodic wave can be written as $J\mathcal{L}$, where J is a skew symmetric matrix and $\mathcal{L}$ is the self-adjoint linearization of the action of the wave (see Section 4 for details). The operator $\mathcal{L}$ is made of two Lamé operators and we are able to calculate the bottom of the spectrum for these operators. To obtain Theorem 1.2, we decompose the space of periodic functions into invariant subspaces: half-periodic and half-anti-periodic, even and odd. Then we analyse the linearized spectrum in each of these subspaces. In the subspace of half-anti-periodic functions, we obtain spectral stability as a consequence of the analysis of the spectrum of $\mathcal{L}$ (alternately, as a consequence of the variational characterizations of Section 3). For the subspace of half-periodic functions, a more involved argument is required. We give in Lemma 4.12 an abstract argument relating coercivity of the linearized action $\mathcal{L}$ with the number of eigenvalues with negative Krein signature of $J\mathcal{L}$ (this is in fact a simplified version of a more general argument [20]). Since we are able to find an eigenvalue with negative Krein signature for $J\mathcal{L}$, spectral stability for half-periodic functions follows from this abstract argument.

Our third main result makes rigorous a formal asymptotic calculation of Rowlands [30] which establishes:

Theorem 1.3. *Cnoidal waves are unstable against perturbations whose period is a sufficiently large multiple of its own.*

This is stated more precisely in Theorem 5.3, and is a consequence of a more general perturbation result, Proposition 5.4, which implies this instability for any real periodic wave for which a certain quantity has the right sign. In particular, the argument does not rely on any integrability (beyond the ability to calculate the quantity in question in terms of elliptic integrals).

Perturbation argument were also used by [14], [15], but our strategy here is different. Instead of relying on abstract theory to obtain the a priori existence of branches of eigenvalues, we directly construct the branch in which we are interested. This is done by first calculating the exact terms of the formal expansion for the eigenvalue and eigenvector at the two first orders, and then obtaining the rigorous existence for the rest of the expansion using a contraction mapping argument. Note that the branch that we are constructing was described in terms of Evans function in [21].

Finally, we complete our analytical results with some numerical observations. Our motivation is to complete the variational characterizations of periodic waves, which was only partial for snoidal waves. We observe:

Observation 1.4. *Let $b < 0$. For a given period, the unique (up to phase shift and translation) global minimizer of the energy with fixed mass and 0 momentum among half-anti-periodic functions is a (appropriately rescaled) snoidal function.*

We have developed a numerical method to obtain the profile ϕ as minimizer on two constraints, fixed mass and fixed (zero) momentum. We use a heat flow algorithm, where at each time step the solution is renormalized to satisfy the constraints. Mass renormalization is simply obtained by scaling. Momentum renormalization is much trickier. We define an auxiliary evolution problem for the momentum that we solve explicitly, and plug back the solution we obtain to get the desired renormalized solutions. We first have tested our algorithm in the cases where our theoretical results hold and we have a good agreement between the theoretical results and the numerical experiments. Then, we have performed experiments on snoidal waves which led to Observation 1.4.

The rest of this paper is divided as follows. In Section 2, we present the spaces of periodic functions and briefly recall the main definitions and properties of Jacobi elliptic functions and integrals. In Section 3, we characterize the Jacobi elliptic functions as global constraint minimizers and give the corresponding orbital stability results. Section 4 is devoted to the proof of spectral stability for cnoidal and snoidal waves, whereas in Section 5 we prove the linear instability of cnoidal waves. Finally, we present our numerical method in Section 6 and the numerical experiments in Section 7.

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2. Preliminaries

This section is devoted to reviewing the classification of real-valued periodic waves in terms of Jacobi elliptic functions.

2.1. **Spaces of Periodic Functions.**

Let $T > 0$ be a period. Denote by τ_T the translation operator

$$(\tau_T f)(x) = f(x+T),$$

acting on $L^2_{\mathrm{loc}}(\mathbb{R})$, and its eigenspaces

$$P_T(\mu) = \{f \in L^2_{\mathrm{loc}}(\mathbb{R}) : \tau_T f = \mu f\}$$

for $\mu \in \mathbb{C}$ with $|\mu| = 1$. Taking $\mu = 1$ yields the space of T-periodic functions

$$P_T = P_T(1) = \{f \in L^2_{\mathrm{loc}}(\mathbb{R}) : \tau_T f = f\},$$

while for $\mu = -1$ we get the T-anti-periodic functions
$$A_T = P_T(-1) = \{f \in L^2_{\mathrm{loc}}(\mathbb{R}) : \tau_T f = -f\}.$$
For $2 \le k \in \mathbb{N}$, letting μ run through the kth roots of unity: $\omega^k = 1$, and $\omega^j \neq 1$ for $1 \le j < k$, we have
$$P_{kT} = \bigoplus_{j=0}^{k-1} P_T(\omega^j),$$
where the decomposition of $f \in P_{kT}$ is given by
$$f = \sum_{j=0}^{k-1} f_j, \quad f_j = \frac{1}{k}\sum_{m=0}^{k-1} \omega^{-mj}\tau_{mT} f.$$
Only the case $k = 2$ is needed here:
$$P_{2T} = P_T \oplus A_T, \qquad f = \frac{1}{2}(f + \tau_T f) + \frac{1}{2}(f - \tau_T f). \tag{2.1}$$
Since the reflection $R : f(x) \mapsto f(-x)$ commutes with τ_T on P_{2T}, we may further decompose into odd and even components in the usual way
$$f = f^+ + f^-, \quad f^\pm = \frac{1}{2}(f \pm Rf),$$
to obtain
$$P_T = P_T^+ \oplus P_T^-, \;\; A_T = A_T^+ \oplus A_T^-, \quad P_T^\pm\ (A_T^\pm) = \{f \in P_T\ (A_T)\ |\ f(-x) = \pm f(x)\},$$
and so
$$P_{2T} = P_T \oplus A_T = P_T^+ \oplus P_T^- \oplus A_T^+ \oplus A_T^-. \tag{2.2}$$
Each of these subspaces is invariant under (1.1), since
$$\psi \in P_T^\pm\ (A_T^\pm) \implies |\psi|^2 \in P_T^+ \implies \psi_{xx} + b|\psi|^2\psi \in P_T^\pm\ (A_T^\pm).$$
When dealing with functions in P_T, we will denote norms such as $L^q(0,T)$ by
$$\|u\|_{L^q} = \|u\|_{L^q(0,T)} = \left(\int_0^T |u|^q\right)^{\frac{1}{q}},$$
and the *complex* L^2 inner product by
$$(f,g) = \int_0^T f\bar{g}\,dx. \tag{2.3}$$

2.2. **Jacobi Elliptic Functions.** Here we recall the definitions and main properties of the Jacobi elliptic functions. The reader might refer to treatises on elliptic functions (e.g. [24]) or to the classical handbooks [1, 17] for more details.

Given $k \in (0,1)$, the *incomplete elliptic integral of the first kind* in trigonometric form is
$$x = F(\phi,k) := \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2\sin^2(\theta)}},$$
and the *Jacobi elliptic functions* are defined through the inverse of $F(\cdot,k)$:
$$\mathrm{sn}(x,k) := \sin(\phi), \quad \mathrm{cn}(x,k) := \cos(\phi), \quad \mathrm{dn}(x,k) := \sqrt{1 - k^2\sin^2(\phi)}.$$
The relations
$$1 = \mathrm{sn}^2 + \mathrm{cn}^2 = k^2\,\mathrm{sn}^2 + \mathrm{dn}^2 \tag{2.4}$$

follow. For extreme value $k = 0$ we recover trigonometric functions,

$$\mathrm{sn}(x,0) = \sin(x), \quad \mathrm{cn}(x,0) = \cos(x), \quad \mathrm{dn}(x,0) = 1,$$

while for extreme value $k = 1$ we recover hyperbolic functions:

$$\mathrm{sn}(x,1) = \tanh(x), \quad \mathrm{cn}(x,1) = \mathrm{dn}(x,1) = \mathrm{sech}(x).$$

The periods of the elliptic functions can be expressed in terms of the *complete elliptic integral of the first kind*

$$K(k) := F\left(\frac{\pi}{2}, k\right), \quad K(k) \to \begin{cases} \frac{\pi}{2} & k \to 0 \\ \infty & k \to 1 \end{cases} .$$

The functions sn and cn are $4K$-periodic while dn is $2K$-periodic. More precisely,

$$\mathrm{dn} \in P^+_{2K}, \quad \mathrm{sn} \in A^-_{2K} \subset P_{4K}, \quad \mathrm{cn} \in A^+_{2K} \subset P_{4K}.$$

The derivatives (with respect to x) of elliptic functions can themselves be expressed in terms of elliptic functions. For fixed $k \in (0,1)$, we have

$$\partial_x \mathrm{sn} = \mathrm{cn} \cdot \mathrm{dn}, \quad \partial_x \mathrm{cn} = -\mathrm{sn} \cdot \mathrm{dn}, \quad \partial_x \mathrm{dn} = -k^2 \mathrm{cn} \cdot \mathrm{sn}, \tag{2.5}$$

from which one can easily verify that sn, cn and dn are solutions of

$$u_{xx} + au + b|u|^2 u = 0, \tag{2.6}$$

with coefficients $a, b \in \mathbb{R}$ for $k \in (0,1)$ given by

$$a = 1 + k^2, \qquad b = -2k^2, \qquad \text{for } u = \mathrm{sn}, \tag{2.7}$$

$$a = 1 - 2k^2, \qquad b = 2k^2, \qquad \text{for } u = \mathrm{cn}, \tag{2.8}$$

$$a = -(2 - k^2), \qquad b = 2, \qquad \text{for } u = \mathrm{dn}. \tag{2.9}$$

2.3. **Elliptic Integrals.** For $k \in (0,1)$, the *incomplete elliptic integral of the second kind* in trigonometric form is defined by

$$E(\phi, k) := \int_0^\phi \sqrt{1 - k^2 \sin^2(\theta)} d\theta.$$

The *complete elliptic integral of the second kind* is defined as

$$E(k) := E\left(\frac{\pi}{2}, k\right).$$

We have the relations (using $d\theta = \mathrm{dn}(z,k)dz$ and $x = F(\phi, k)$)

$$\begin{aligned} E(\phi, k) &= \int_0^x \mathrm{dn}^2(z,k)dz \\ &= x - k^2 \int_0^x \mathrm{sn}^2(z,k)dz = (1-k^2)x + k^2 \int_0^x \mathrm{cn}^2(z,k)dz, \end{aligned} \tag{2.10}$$

relating the elliptic functions to the elliptic integral of the second kind, and

$$E(k) = K(k) - k^2 \int_0^K \mathrm{sn}^2(z,k)dz = (1-k^2)K(k) + k^2 \int_0^K \mathrm{cn}^2(z,k)dz, \tag{2.11}$$

relating the elliptic integrals of first and second kind. We can differentiate E and K with respect to k and express the derivatives in terms of E and K:

$$\partial_k E(k) = \frac{E(k) - K(k)}{k} < 0,$$
$$\partial_k K(k) = \frac{E(k) - (1-k^2)K(k)}{k - k^3} = \frac{k^2 \int_0^K \operatorname{cn}^2(x,k)dx}{k - k^3} > 0.$$

Note in particular K is increasing, E is decreasing. Moreover,

$$K(0) = E(0) = \frac{\pi}{2}, \quad K(1-) = \infty, \quad E(1) = 1.$$

2.4. **Classification of Real Periodic Waves.** Here we make precise the fact that the elliptic functions provide the only (non-constant) real-valued, periodic solutions of (2.6). Note that there is a two-parameter family of complex-valued, bounded, solutions for every $a, b \in \mathbb{R}$, $b \neq 0$ [12, 14].

Lemma 2.1 (focusing case). *Fix a period $T > 0$, $a \in \mathbb{R}$, $b > 0$ and $u \in P_T$ a non-constant real solution of* (2.6). *By invariance under translation, and negation ($u \mapsto -u$), we may suppose $u(0) = \max u > 0$.*

(a) *If $0 \le \min u < u(0)$, then $a < 0$, $|a| < bu(0)^2 < 2|a|$, and $u(x) = \frac{1}{\alpha}\operatorname{dn}(\frac{x}{\beta}, k)$,*

(b) *If $\min u < 0$, then $\max(0, -2a) < bu(0)^2$, and $u(x) = \frac{1}{\alpha}\operatorname{cn}(\frac{x}{\beta}, k)$,*

for some $\alpha > 0$, $\beta > 0$, and $0 < k < 1$, uniquely determined by T, a, b and $\max u$. They satisfy the a-independent relations $b\beta^2 = 2\alpha^2$ for (a) *and $b\beta^2 = 2k^2\alpha^2$ for* (b).

Note that here T may be any multiple of the fundamental period of u. An a-independent relation is useful since a will be the unknown Lagrange multiplier for our constrained minimization problems in Section 3.

Proof. The first integral is constant: there exists $C_0 \in \mathbb{R}$ such that

$$u_x^2 + au^2 + \frac{b}{2}u^4 = C_0.$$

A periodic solution has to oscillate in the energy well $W(u) = au^2 + \frac{b}{2}u^4$ with energy level C_0. If $0 \le \min u$, then $a < 0$ and $C_0 < 0$. If $\min u < 0$, then $C_0 > 0$. Let $u(x) = \frac{1}{\alpha}v(\frac{x}{\beta})$ with $\alpha = (\max u)^{-1}$. Then v satisfies

$$\max v = v(0) = 1, \quad v'' + a\beta^2 v + \frac{b\beta^2}{\alpha^2}v^3 = 0.$$

(a) If $0 \le \min u$, then $a < 0$ and $C_0 < 0$. Let $0 < y_1 < y_2$ be the roots of $ay + \frac{b}{2}y^2 = C_0 < 0$. Then $u(0)^2 = y_2 \in (-a/b, -2a/b)$.

Let $\beta = \alpha\sqrt{2/b}$. Then $\frac{b\beta^2}{\alpha^2} = 2$ and $a\beta^2 \in (-2,-1)$, and there is a unique $k \in (0,1)$ so that $a\beta^2 = -2 + k^2$. Thus

$$\max v = v(0) = 1, \quad v'(0) = 0, \quad v'' + (-2+k^2)v + 2v^3 = 0.$$

By uniqueness of the ODE, $v(x) = \operatorname{dn}(x,k)$ is the only solution. Hence $u(x) = \frac{1}{\alpha}\operatorname{dn}(\frac{x}{\beta}, k)$.

(b) If $\min u < 0$, then $C_0 > 0$. Let $y_1 < 0 < y_2$ be the roots of $ay + \frac{b}{2}y^2 = C_0 > 0$. Then $u(0)^2 = y_2 > \max(0, -2a/b)$ no matter $a < 0$ or $a \ge 0$. We claim we can choose unique $\beta > 0$ and $k \in (0,1)$ so that

$$a\beta^2 = 1 - 2k^2, \quad \frac{b\beta^2}{\alpha^2} = 2k^2.$$

The sum gives $(a + \frac{b}{\alpha^2})\beta^2 = 1$, thus $\beta = (a + \frac{b}{\alpha^2})^{-1/2}$ noting $(a + \frac{b}{\alpha^2}) > 0$, and

$$k^2 = \frac{\beta^2 b}{2\alpha^2} = \frac{b}{2(b + a\alpha^2)} \in (0,1)$$

no matter $a < 0$ or $a \geq 0$. Thus

$$\max v = v(0) = 1, \quad v'(0) = 0, \quad v'' + (1 - 2k^2)v + 2k^2 v^3 = 0.$$

By uniqueness of the ODE, $v(x) = \operatorname{cn}(x, k)$ is the only solution. Hence $u(x) = \frac{1}{\alpha} \operatorname{cn}(\frac{x}{\beta}, k)$. □

Lemma 2.2 (defocusing case)**.** *Fix a period $T > 0$, $a \in \mathbb{R}$, $b < 0$ and $u \in P_T$ a non-constant, real solution of* (2.6)*. By invariance under translation and negation, suppose $u(0) = \max u > 0$. Then $0 < |b|u(0)^2 < a$, and $u(x) = \frac{1}{\alpha} \operatorname{sn}(K(k) + \frac{x}{\beta}, k)$, for some $\alpha > 0$, $\beta > 0$, and $0 < k < 1$, uniquely determined by T, a, b and $\max u$. They satisfy the a-independent relation $b\beta^2 = -2k^2\alpha^2$.*

Proof. The first integral is constant: there exists $C_0 \in \mathbb{R}$ such that

$$u_x^2 + au^2 + \frac{b}{2}u^4 = C_0.$$

A periodic solution has to oscillate in the energy well $W(u) = au^2 + \frac{b}{2}u^4$ with energy level C_0. Hence $a > 0$ and $0 < C_0 < \max W = \frac{a^2}{-2b}$. Let $u(x) = \frac{1}{\alpha} v(\frac{x}{\beta})$ with $\alpha = (\max u)^{-1}$. Then v satisfies

$$\max v = v(0) = 1, \quad v'' + a\beta^2 v + \frac{b\beta^2}{\alpha^2} v^3 = 0.$$

Let $0 < y_1 < y_2$ be the roots of $ay + \frac{b}{2}y^2 = C_0$. Then $u(0)^2 = y_1 \in (0, -a/b)$.

Let $\beta = (\frac{2\alpha^2}{2\alpha^2 a + b})^{1/2}$ and $k = (\frac{-b}{2\alpha^2 a + b})^{1/2}$, noting $2\alpha^2 a + b > 0$. Then $a\beta^2 = 1 + k^2$, $\frac{b\beta^2}{\alpha^2} = -2k^2$, and v satisfies

$$\max v = v(0) = 1, \quad v'(0) = 0, \quad v'' + (1 + k^2)v - 2k^2 v^3 = 0.$$

By uniqueness of the ODE, $v(x) = \operatorname{sn}(K(k) + x, k)$ is the only solution. Hence $u(x) = \frac{1}{\alpha} \operatorname{sn}(K(k) + \frac{x}{\beta}, k)$. □

3. Variational Characterizations and Orbital Stability

Our goal in this section is to characterize the Jacobi elliptic functions as *global* constrained energy minimizers. As a corollary, we recover some known results on orbital stability, which is closely related to *local* variational information.

3.1. **The Minimization Problems.** Recall the basic conserved functionals for (1.1) on $H^1_{\mathrm{loc}} \cap P_T$:

$$\mathcal{M}(u) = \frac{1}{2}\int_0^T |u|^2 dx, \quad \mathcal{P}(u) = \frac{1}{2}\mathcal{Im}\int_0^T u\bar{u}_x dx,$$
$$\mathcal{E}(u) = \frac{1}{2}\int_0^T |u_x|^2 dx - \frac{b}{4}\int_0^T |u|^4 dx.$$

In this section, we consider $L^2(0,T;\mathbb{C})$ as a *real* Hilbert space with scalar product $\mathcal{Re}\int_0^T f\bar{g}dx$. This way, the functionals $\mathcal{E}$, $\mathcal{M}$ and $\mathcal{P}$ are C^1 functionals. This also ensures that the Lagrange multipliers are real. Note that we see $L^2(0,T;\mathbb{C})$ as a real Hilbert space only in the current section and in all the other sections it will be seen as a complex Hilbert space with the scalar product defined in (2.3).

Fix parameters $T > 0$, $a, b \in \mathbb{R}$, $b \neq 0$. Since the Jacobi elliptic functions (indeed any standing wave profiles) are solutions of (2.6), they are critical points of the *action* functional $\mathcal{S}_a$ defined by

$$\mathcal{S}_a(u) = \mathcal{E}(u) - a\mathcal{M}(u),$$

where the values of a and b are given in (2.7)-(2.9) and the fundamental periods are $T = 2K$ for dn, $T = 4K$ for sn, cn. Given $m > 0$, the basic variational problem is to minimize the energy with fixed mass:

$$\min\left\{\mathcal{E}(u) \mid \mathcal{M}(u) = m, u \in H^1_{\mathrm{loc}} \cap P_T\right\}, \tag{3.1}$$

whose Euler-Lagrange equation

$$u'' + b|u|^2 u + au = 0, \tag{3.2}$$

with $a \in \mathbb{R}$ arising as Lagrange multiplier, is indeed of the form (2.6). Since the momentum is also conserved for (1.1), it is natural to consider the problem with a further momentum constraint:

$$\min\left\{\mathcal{E}(u) \mid \mathcal{M}(u) = m, \mathcal{P}(u) = 0, u \in H^1_{\mathrm{loc}} \cap P_T\right\}. \tag{3.3}$$

Remark 3.1. Note that if a minimizer u of (3.1) is such that $P(u) = 0$, then it is real-valued (up to multiplication by a complex number of modulus 1). Indeed, it verifies (3.2) for some $a \in \mathbb{R}$. It is well known (see e.g. [13]) that the momentum density $\mathcal{Im}(u_x\bar{u})$ is therefore constant in x, and so it is identically 0 if $P(u) = 0$. For $u(x) \neq 0$ we can write u as $u = \rho e^{i\theta}$, and express the momentum density as $\mathcal{Im}(u_x\bar{u}) = \theta_x\rho^2$. Thus $\mathcal{Im}(u_x\bar{u}) = 0$ implies $\theta_x = 0$ and thus $\theta(x)$ is constant as long as $u(x) \neq 0$. If $u(x_0) = 0$ and $e^{\theta(x_0-)} \neq e^{\theta(x_0+)}$, we must have $u_x(x_0) = 0$, and hence $u \equiv 0$ by uniqueness of the ODE.

Since (1.1) preserves the subspaces in the decomposition (2.1), it is also natural to consider variational problems restricted to anti-symmetric functions,

$$\min\left\{\mathcal{E}(u) \mid \mathcal{M}(u) = m, u \in H^1_{\mathrm{loc}} \cap A_{T/2}\right\}, \tag{3.4}$$
$$\min\left\{\mathcal{E}(u) \mid \mathcal{M}(u) = m, \mathcal{P}(u) = 0, u \in H^1_{\mathrm{loc}} \cap A_{T/2}\right\}, \tag{3.5}$$

and in light of the decomposition (2.2), further restrictions to even or odd functions may also be considered.

In general, the difficulty does not lie in proving the existence of a minimizer, but rather in identifying this minimizer with an elliptic function, since we are minimizing among *complex valued* functions, and moreover restrictions to symmetry subspaces prevent us from using classical variational methods like symmetric rearrangements.

We will first consider the minimization problems (3.1) and (3.3) for periodic functions in P_T. Then we will consider the minimization problems (3.4) and (3.5) for half-anti-periodic functions in $A_{T/2}$. In both parts, we will treat separately the focusing ($b>0$) and defocusing ($b<0$) nonlinearities. For each case, we will show the existence of a unique (up to phase shift and translation) minimizer, and we will identify it with either a plane wave or a Jacobi elliptic function.

3.2. **Minimization Among Periodic Functions.**

3.2.1. *The Focusing Case in P_T.*

Proposition 3.2. *Assume $b>0$. The minimization problems (3.1) and (3.3) satisfy the following properties.*

(i) *For all $m>0$, (3.1) and (3.3) share the same minimizers. The minimal energy is finite and negative.*

(ii) *For all $0<m\leqslant \frac{\pi^2}{bT}$ there exists a unique (up to phase shift) minimizer of (3.1). It is the constant function $u_{\min}\equiv\sqrt{\frac{2m}{T}}$.*

(iii) *For all $\frac{\pi^2}{bT}<m<\infty$ there exists a unique (up to translations and phase shift) minimizer of (3.1). It is the rescaled function $\mathrm{dn}_{\alpha,\beta,k}=\frac{1}{\alpha}\,\mathrm{dn}\left(\frac{\cdot}{\beta},k\right)$ where the parameters α, β and k are uniquely determined. Its fundamental period is T. The map from $m\in(\frac{\pi^2}{bT},\infty)$ to $k\in(0,1)$ is one-to-one, onto and increasing.*

(iv) *In particular, given $k\in(0,1)$, $\mathrm{dn}=\mathrm{dn}(\cdot,k)$, if $b=2$, $T=2K(k)$, and $m=\mathcal{M}(\mathrm{dn})=E(k)$, then the unique (up to translations and phase shift) minimizer of (3.1) is dn.*

Proof. Without loss of generality, we can restrict the minimization to real-valued non-negative functions. Indeed, if $u\in H^1_{\mathrm{loc}}\cap P_T$, then $|u|\in H^1_{\mathrm{loc}}\cap P_T$ and we have

$$\|\partial_x|u|\|_{L^2}\leqslant\|\partial_x u\|_{L^2}.$$

This readily implies that (3.1) and (3.3) share the same minimizers. Let us prove that

$$-\infty<\min\left\{\mathcal{E}(u)\mid\mathcal{M}(u)=m,u\in H^1_{\mathrm{loc}}\cap P_T\right\}<0. \tag{3.6}$$

The last inequality in (3.6) is obtained using the constant function $\varphi_{m,0}\equiv\sqrt{\frac{2m}{T}}$ as a test function:

$$\mathcal{E}(\varphi_{m,0})<0,\quad \mathcal{M}(\varphi_{m,0})=m.$$

To prove the first inequality in (3.6), we observe that by Gagliardo-Nirenberg inequality we have

$$\|u\|_{L^4}^4\lesssim\|u\|_{L^2}^3\|u_x\|_{L^2}+\|u\|_{L^2}^4.$$

Consequently, for $u\in H^1_{\mathrm{loc}}\cap P_T$ such that $\mathcal{M}(u)=m$, we have

$$\mathcal{E}(u)\gtrsim\|u_x\|_{L^2}\left(\|u_x\|_{L^2}-m^{3/2}\right)-m^2,$$

and $\mathcal{E}$ has to be bounded from below. The above shows (i).

Consider now a minimizing sequence $(u_n)\subset H^1_{\mathrm{loc}}\cap P_T$ for (3.1). It is bounded in $H^1_{\mathrm{loc}}\cap P_T$ and therefore, up to a subsequence, it converges weakly in $H^1_{\mathrm{loc}}\cap P_T$ and strongly in $L^2_{\mathrm{loc}}\cap P_T$ and $L^4_{\mathrm{loc}}\cap P_T$ towards $u_\infty\in H^1_{\mathrm{loc}}\cap P_T$. Therefore

$\mathcal{E}(u_\infty) \leqslant \mathcal{E}(u_n)$ and $\mathcal{M}(u_\infty) = m$. This implies that $\|\partial_x u_\infty\|_{L^2} = \lim_{n\to\infty} \|\partial_x u_n\|_{L^2}$ and therefore the convergence from u_n to u_∞ is also strong in $H^1_{\mathrm{loc}} \cap P_T$. Since u_∞ is a minimizer of (3.1), there exists a Lagrange multiplier $a \in \mathbb{R}$ such that

$$-\mathcal{E}'(u_\infty) + a\mathcal{M}'(u_\infty) = 0,$$

that is

$$\partial_{xx} u_\infty + b u_\infty^3 + a u_\infty = 0.$$

Multiplying by u_∞ and integrating, we find that

$$a = \frac{\|\partial_x u_\infty\|_{L^2}^2 - b\|u_\infty\|_{L^4}^4}{\|u_\infty\|_{L^2}^2}.$$

Note that

$$\|\partial_x u_\infty\|_{L^2}^2 - b\|u_\infty\|_{L^4}^4 = 2\mathcal{E}(u_\infty) - \frac{b}{2}\|u_\infty\|_{L^4}^4 < 0,$$

therefore

$$a < 0.$$

We already have $u_\infty \in \mathbb{R}$, and we may assume $\max u = u(0)$ by translation. By Lemma 2.1 (a), either u_∞ is constant or there exist $\alpha, \beta \in (0,\infty)$ and $k \in (0,1)$ such that $\beta = \alpha\sqrt{2/b}$ and

$$u_\infty(x) = \mathrm{dn}_{\alpha,\beta,k}(x) = \frac{1}{\alpha}\,\mathrm{dn}\left(\frac{x}{\beta}, k\right).$$

We now show that the minimizer u_∞ is of the form $\mathrm{dn}_{\alpha,\beta,k}$ if $m > \frac{\pi^2}{bT}$. Indeed, assuming by contradiction that u_∞ is a constant, we necessarily have $u_\infty \equiv \sqrt{\frac{2m}{T}}$. The Lagrange multiplier can also be computed and we find $a = -bu_\infty^2 = -\frac{2bm}{T}$. Since u_∞ is supposed to be a constrained minimizer for (3.1), the operator

$$-\partial_{xx} - a - 3bu_\infty^2 = -\partial_{xx} - \frac{4bm}{T}$$

must have Morse index at most 1, i.e. at most 1 negative eigenvalue. The eigenvalues are given for $n \in \mathbb{Z}$ by the formula

$$\left(\frac{2\pi n}{T}\right)^2 - \frac{4bm}{T}.$$

Obviously $n = 0$ gives a negative eigenvalue. For $n = 1$, the eigenvalue is non-negative if and only if

$$m \leqslant \frac{\pi^2}{bT},$$

which gives the contradiction. Hence when $m > \frac{\pi^2}{bT}$ the minimizer u_∞ must be of the form $\mathrm{dn}_{\alpha,\beta,k}$.

There is a positive integer n so that the fundamental period of $u_\infty = \mathrm{dn}_{\alpha,\beta,k}$ is $2K(k)\beta = Tn^{-1}$. As already mentioned, since u_∞ is a minimizer for (3.1), the operator

$$-\partial_{xx} - a - 3bu_\infty^2$$

can have at most one negative eigenvalue. The function $\partial_x u_\infty$ is in its kernel and has $2n$ zeros. By Sturm-Liouville theory (see e.g. [10, 29]) we have at least $2n-1$ eigenvalues below 0. Hence $n = 1$ and $2K(k)\beta = T$.

Using $2\alpha^2 = b\beta^2$ (see Lemma 2.1), the mass verifies,

$$m = \frac{1}{2}\int_0^T |\operatorname{dn}_{\alpha,\beta,k}(x)|^2 dx = \frac{\beta}{\alpha^2}\frac{1}{2}\int_0^{2K(k)} |\operatorname{dn}(y,k)|^2 dy = \frac{2}{b\beta}E(k)$$

where $E(k)$ is given in Section 2.3. Using $2K(k)\beta = T$,

$$m = \frac{4}{bT}E(k)K(k). \tag{3.7}$$

Note

$$\frac{\partial}{\partial k}EK(k) = \frac{E(k)^2 - (1-k^2)K(k)^2}{(1-k^2)k} > 0,$$

where the positivity of the numerator is because it vanishes at $k = 0$ and

$$\frac{\partial}{\partial k}(E^2 - (1-k^2)K^2) = \frac{2}{k}(E-K)^2, \quad (0 < k < 1).$$

Thus $EK(k)$ varies from $\frac{\pi^2}{4}$ to ∞ when k varies from 0 to 1. Thus (3.7) defines m as a strictly increasing function of $k \in (0,1)$ with range $(\frac{\pi^2}{bT}, \infty)$ and hence has an inverse function. For fixed b, m, T, the value $k \in (0,1)$ is uniquely determined by (3.7). We also have $\beta = \frac{T}{2K(k)}$ and $\alpha = \beta\sqrt{b/2}$. The above shows (iii).

The above calculation also shows that $m > \frac{\pi^2}{bT}$ if $u_\infty = \operatorname{dn}_{\alpha,\beta,k}$. Thus u_∞ must be a constant when $0 < m \leq \frac{\pi^2}{bT}$. This shows (ii).

In the case we are given $k \in (0,1)$, $T = 2K(k)$, $b = 2$ and $m = \mathcal{M}(\operatorname{dn}) = E(k)$, we want to show that $u_\infty(x) = \operatorname{dn}(x,k)$. In this case $m > \frac{\pi^2}{bT}$ since $EK > \frac{\pi^2}{4}$. Thus, by Lemma 2.1 (a), $u_\infty = \operatorname{dn}_{\alpha,\beta,s}$ for some $\alpha, \beta > 0$ and $s \in (0,1)$, up to translation and phase. By the same Sturm-Liouville theory argument, the fundamental period of u_∞ is $T = 2K(s)\beta$. The same calculation leading to (3.7) shows

$$m = \frac{4}{bT}E(s)K(s).$$

Thus $E(k)K(k) = E(s)K(s)$. Using the monotonicity of $EK(k)$ in k, we have $k = s$. Thus $\alpha = \beta = 1$ and $u_\infty(x) = \operatorname{dn}(x,k)$. This gives (iv) and finishes the proof. $\square$

3.2.2. *The Defocusing Case in P_T.*

Proposition 3.3. *Assume $b < 0$. For all $0 < m < \infty$, the constrained minimization problems (3.1) and (3.3) have the same unique (up to phase shift) minimizers, which is the constant function $u_{\min} \equiv \sqrt{\frac{2m}{T}}$.*

Proof. This is a simple consequence of the fact that functions with constant modulus are the optimizers of the injection $L^4(0,T) \hookrightarrow L^2(0,T)$. More precisely, for every $f \in L^4(0,T)$ we have by Hölder's inequality,

$$\|f\|_{L^2} \leqslant T^{1/4}\|f\|_{L^4},$$

with equality if and only if $|f|$ is constant. Let $\varphi_{m,0}$ be the constant function $\varphi_{m,0} \equiv \sqrt{\frac{2m}{T}}$. For any $v \in H^1_{\text{loc}} \cap P_T$ such that $\mathcal{M}(v) = m$ and $v \not\equiv e^{i\theta}\varphi_{m,0}$ ($\theta \in \mathbb{R}$) we have

$$0 = \|\partial_x \varphi_{m,0}\|_{L^2}^2 < \|\partial_x v\|_{L^2}^2,$$
$$\|\varphi_{m,0}\|_{L^4}^4 = 4T^{-1}\mathcal{M}^2(\varphi_{m,0}) = 4T^{-1}\mathcal{M}^2(v) \leqslant \|v\|_{L^4}^4.$$

As a consequence, $\mathcal{E}(\varphi_{m,0}) < \mathcal{E}(v)$ and this proves the proposition. □

3.3. **Minimization Among Half-Anti-Periodic Functions.**

3.3.1. *The Focusing Case in $A_{T/2}$.*

Proposition 3.4. *Assume $b > 0$. For all $m > 0$, the minimization problems (3.4) and (3.5) in $A_{T/2}$ satisfy the following properties.*

(i) *The minimizers for (3.4) and (3.5) are the same.*
(ii) *There exists a unique (up to translations and phase shift) minimizer of (3.4). It is the rescaled function $\mathrm{cn}_{\alpha,\beta,k} = \frac{1}{\alpha}\,\mathrm{cn}\left(\frac{\cdot}{\beta}, k\right)$ where the parameters α, β and k are uniquely determined. Its fundamental period is T. The map from $m \in (0,\infty)$ to $k \in (0,1)$ is one-to-one, onto and increasing.*
(iii) *In particular, given $k \in (0,1)$, $\mathrm{cn} = \mathrm{cn}(\cdot,k)$, if $b = 2k^2$, $T = 4K(k)$, and $m = \mathcal{M}(\mathrm{cn}) = 2(E-(1-k^2)K)/k^2$, then the unique (up to translations and phase shift) minimizer of (3.4) is* cn.

Before proving Proposition 3.4, we make the following crucial observation.

Lemma 3.5. *Let $v \in H^1_{\mathrm{loc}} \cap A_{T/2}$. Then there exists $\tilde{v} \in H^1_{\mathrm{loc}} \cap A_{T/2}$ such that*

$$\tilde{v}(x) \in \mathbb{R}, \quad \|\tilde{v}\|_{L^2} = \|v\|_{L^2}, \quad \|\partial_x \tilde{v}\|_{L^2} = \|\partial_x v\|_{L^2}, \quad \|\tilde{v}\|_{L^4} \geqslant \|v\|_{L^4}.$$

Proof of Lemma 3.5. The proof relies on a combinatorial argument. Since $v \in H^1_{\mathrm{loc}} \cap A_{T/2}$, its Fourier series expansion contains only terms indexed by *odd* integers:

$$v(x) = \sum_{\substack{j\in\mathbb{Z}\\ j \text{ odd}}} v_j e^{ij\frac{2\pi}{T}x}.$$

We define $\tilde{v}$ by its Fourier series expansion

$$\tilde{v}(x) = \sum_{\substack{j\in\mathbb{Z}\\ j \text{ odd}}} \tilde{v}_j e^{ij\frac{2\pi}{T}x}, \quad \tilde{v}_j := \sqrt{\frac{|v_j|^2 + |v_{-j}|^2}{2}}.$$

It is clear that $\tilde{v}(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$, and by Plancherel formula,

$$\|\tilde{v}\|_{L^2} = \|v\|_{L^2}, \quad \|\partial_x \tilde{v}\|_{L^2} = \|\partial_x v\|_{L^2},$$

so all we have to prove is that $\|\tilde{v}\|_{L^4} \geqslant \|v\|_{L^4}$. We have

$$|v(x)|^2 = \sum_{\substack{j\in\mathbb{Z}\\ j \text{ odd}}} |v_j|^2 + \sum_{\substack{n\in 2\mathbb{N}\\ n\geqslant 2}} w_n e^{in\frac{2\pi}{T}x} + \bar{w}_n e^{-in\frac{2\pi}{T}x},$$

where we have defined

$$w_n = \sum_{\substack{j>k, j+k=n\\ j,k \text{ odd}}} v_j \bar{v}_{-k} + v_k \bar{v}_{-j}.$$

Using the fact that for $n \in \mathbb{N}$, $n \neq 0$, the term $e^{in\frac{2\pi}{T}x}$ integrates to 0 due to periodicity,

$$\int_0^T e^{in\frac{2\pi}{T}x} dx = 0,$$

we compute

$$\frac{1}{T}\int_0^T |v|^4 dx = \Bigg(\sum_{\substack{j\in\mathbb{Z} \\ j \text{ odd}}} |v_j|^2 \Bigg)^2 + 2 \sum_{\substack{n\in 2\mathbb{N} \\ n\geqslant 2}} |w_n|^2.$$

The first part is just

$$\Bigg(\sum_{\substack{j\in\mathbb{Z} \\ j \text{ odd}}} |v_j|^2 \Bigg)^2 = \frac{1}{T^2}\|v\|_{L^2}^4 = \frac{1}{T^2}\|\tilde{v}\|_{L^2}^4.$$

For the second part, we observe that

$$w_n = \sum_{\substack{j>k, j+k=n \\ j,k \text{ odd}}} \begin{pmatrix} v_j \\ \bar{v}_{-j} \end{pmatrix} \cdot \begin{pmatrix} v_{-k} \\ \bar{v}_k \end{pmatrix}, \tag{3.8}$$

where the $\cdot$ denotes the complex vector scalar product. Therefore,

$$\begin{aligned} |w_n| &\leqslant \sum_{\substack{j>k, j+k=n \\ j,k \text{ odd}}} \left| \begin{pmatrix} v_j \\ \bar{v}_{-j} \end{pmatrix} \right| \left| \begin{pmatrix} v_{-k} \\ \bar{v}_k \end{pmatrix} \right| = \sum_{\substack{j>k, j+k=n \\ j,k \text{ odd}}} \sqrt{2\tilde{v}_j^2}\sqrt{2\tilde{v}_k^2} \\ &= 2 \sum_{\substack{j>k, j+k=n \\ j,k \text{ odd}}} \tilde{v}_j \tilde{v}_k = \tilde{w}_n, \end{aligned}$$

where by $\tilde{w}_n$ we denote the quantity defined similarly as in (3.8) for $(\tilde{v}_j)$. As a consequence,

$$\|v\|_{L^4} \leqslant \|\tilde{v}\|_{L^4}$$

and this finishes the proof of Lemma 3.5. □

Proof of Proposition 3.4. All functions are considered in $A_{T/2}$. Consider a minimizing sequence (u_n) for (3.5). By Lemma 3.5, the minimizing sequence can be chosen such that $u_n(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$ and this readily implies the equivalence between (3.5) and (3.4), which is (i).

Using the same arguments as in the proof of Proposition 3.2, we infer that the minimizing sequence converges strongly in $H^1_{\text{loc}} \cap A_{T/2}$ to $u_\infty \in H^1_{\text{loc}} \cap A_{T/2}$ verifying for some $a \in \mathbb{R}$ the Euler-Lagrange equation

$$\partial_{xx} u_\infty + b u_\infty^3 + a u_\infty = 0.$$

Then, since u_∞ is real and in $A_{T/2}$, we may assume $\max u = u(0) > 0$ and, by Lemma 2.1 (b), there exists a set of parameters $\alpha, \beta \in (0,\infty)$, $k \in (0,1)$ such that

$$u_\infty(x) = \frac{1}{\alpha} \operatorname{cn}\left(\frac{x}{\beta}, k\right),$$

and the parameters α, β, k are determined by T, a, b and $\max u$, with $2k^2\alpha^2 = b\beta^2$.

There exists an odd, positive integer n so that the fundamental period of u_∞ is $4K(k)\beta = T/n$. Since u_∞ is a minimizer for (3.4), the operator

$$-\partial_{xx} - a - 3bu_\infty^2$$

can have at most one negative eigenvalue in $L^2_{\text{loc}} \cap A_{T/2}$. The function $\partial_x u_\infty$ is in its kernel and has $2n$ zeros in $[0,T)$. By Sturm-Liouville theory, there are at least $n-1$ eigenvalues (with eigenfunctions in $A_{T/2}$) below 0. Hence, since n is odd, $n = 1$ and $4K(k)\beta = T$.

The mass verifies, using $2k^2\alpha^2 = b\beta^2$ and (2.11),

$$m = \frac{1}{2}\int_0^T |\operatorname{cn}_{\alpha,\beta,k}(x)|^2 dx = \frac{\beta}{\alpha^2}\frac{1}{2}\int_0^{4K(k)} |\operatorname{cn}(y,k)|^2 dy = \frac{4}{\beta b}(E(k) - (1-k^2)K(k)).$$

Using $4K(k)\beta = T$,

$$m = M(k) := \frac{16}{bT}K(k)(E(k) - (1-k^2)K(k)). \tag{3.9}$$

Note all factors of $M(k)$ are positive, $\frac{\partial}{\partial k}K(k) > 0$ and

$$\frac{\partial}{\partial k}(E - (1-k^2)K) = \frac{E-K}{k} - \frac{E-(1-k^2)K}{k} + 2kK = kK > 0.$$

Thus (3.9) defines m as a strictly increasing function of $k \in (0,1)$ with range $(0,\infty)$ and hence has an inverse function. For fixed T, b, m, the value $k \in (0,1)$ is uniquely determined by (3.9). We also have $\beta = \frac{T}{4K(k)}$ and $\alpha^2 = \frac{b\beta^2}{2k^2}$. The above shows (ii).

In the case we are given $k \in (0,1)$, $T = 4K(k)$, $b = 2k^2$ and $m = \mathcal{M}(\operatorname{cn}(\cdot,k))$, we want to show that $u_\infty(x) = \operatorname{cn}(x,k)$. In this case, by Lemma 2.1 (b), $u_\infty = \operatorname{cn}_{\alpha,\beta,s}$ for some $\alpha, \beta > 0$ and $s \in (0,1)$, up to translation and phase. By the same Sturm-Liouville theory argument, the fundamental period of u_∞ is $T = 4K(s)\beta$. The same calculation leading to (3.9) shows

$$m = M(s).$$

Thus $M(s) = M(k)$. By the monotonicity of $M(k)$ in k, we have $k = s$. Thus $\alpha = \beta = 1$ and $u_\infty(x) = \operatorname{cn}(x,k)$. This shows (iii) and concludes the proof. $\square$

3.3.2. *The Defocusing Case in $A_{T/2}$.*

Proposition 3.6. *Assume $b < 0$. There exists a unique (up to phase shift and complex conjugate) minimizer for (3.4). It is the plane wave $u_{\min} \equiv \sqrt{\frac{2m}{T}}e^{\frac{2i\pi x}{T}}$.*

Proof. Denote the supposed minimizer by $w(x) = \sqrt{\frac{2m}{T}}e^{\pm\frac{2i\pi x}{T}}$. Let $v \in H^1_{\mathrm{loc}} \cap A_{2K}$ such that $\mathcal{M}(v) = m$ and $v \not\equiv e^{i\theta}w$ ($\theta \in \mathbb{R}$). As in the proof of Proposition 3.3, we have

$$\|w\|_{L^4}^4 = 4T^{-1}\mathcal{M}^2(w) = 4T^{-1}\mathcal{M}^2(v) \leqslant \|v\|_{L^4}^4.$$

Since $v \in A_{2K}$, v must have 0 mean value. Recall that in that case v verifies the Poincaré-Wirtinger inequality

$$\|v\|_{L^2} \leqslant \frac{T}{2\pi}\|v'\|_{L^2},$$

and that the optimizers of the Poincaré-Wirtinger inequality are of the form $Ce^{\pm\frac{2i\pi}{T}x}$, $C \in \mathbb{C}$. This implies that

$$\|\partial_x w\|_{L^2}^2 = \frac{8\pi^2}{T^2}\mathcal{M}(w) = \frac{8\pi^2}{T^2}\mathcal{M}(v) < \|\partial_x v\|_{L^2}^2.$$

As a consequence, $\mathcal{E}(w) < \mathcal{E}(v)$ and this proves the lemma. $\square$

As far as (3.5) is concerned, we make the following conjecture

Conjecture 3.7. *Assume $b<0$. The unique (up to translations and phase shift) minimizer of (3.5) is the rescaled function $\mathrm{sn}_{\alpha,\beta,k} = \frac{1}{\alpha}\,\mathrm{sn}\left(\frac{\cdot}{\beta},k\right)$ where the parameters α, β and k are uniquely determined.*

In particular, given $k \in (0,1)$, $\mathrm{sn} = \mathrm{sn}(\cdot,k)$, if $b=-2k^2$, $T=4K(k)$, and $m=\mathcal{M}(\mathrm{sn})$, then the unique (up translations and to phase shift) minimizer of (3.5) is sn.

This conjecture is supported by numerical evidence, see Observation 7.1. The main difficulty in proving the conjecture is to show that the minimizer is real up to a phase.

3.3.3. *The Defocusing Case in $A^-_{T/2}$.* In light of our uncertainty about whether sn solves (3.5), let us settle for the simple observation that it is the energy minimizer among *odd*, half-anti-periodic functions:

Proposition 3.8. *Assume $b<0$. The unique (up to phase shift) minimizer of the problem*

$$\min\left\{\mathcal{E}(u) \mid \mathcal{M}(u)=m, u\in H^1_{\mathrm{loc}}\cap A^-_{T/2}\right\}, \tag{3.10}$$

is the rescaled function $\mathrm{sn}_{\alpha,\beta,k} = \frac{1}{\alpha}\,\mathrm{sn}\left(\frac{\cdot}{\beta},k\right)$ where the parameters α, β and k are uniquely determined. Its fundamental period is T. The map from $m\in(0,\infty)$ to $k\in(0,1)$ is one-to-one, onto and increasing.

In particular, given $k\in(0,1)$, $\mathrm{sn}=\mathrm{sn}(\cdot,k)$, if $b=-2k^2$, $T=4K(k)$, and $m=\mathcal{M}(\mathrm{sn})$, then the unique (up to phase shift) minimizer of (3.10) is sn.

Proof. If $u\in A^-_{T/2}$, then $0=u(0)=u(T/2)$, and since u is completely determined by its values on $[0,T/2]$, we may replace (3.10) by

$$\min\left\{\int_0^{T/2}\left(|u_x|^2-\frac{b}{2}|u|^4\right)dx \mid \int_0^{T/2}|u(x)|^2dx=m,\ u\in H^1_0([0,T])\right\},$$

for which the map $u\mapsto |u|$ is admissible, showing that minimizers are non-negative (up to phase), and in particular real-valued, hence a (rescaled) sn function by Lemma 2.2. The remaining statements follow as in the proof of Proposition 3.4. In particular, the mass verifies, using $2k^2\alpha^2=|b|\beta^2$, (2.11), and $4K(k)\beta=T$,

$$\begin{aligned} m &= \frac{1}{2}\int_0^T|\,\mathrm{sn}_{\alpha,\beta,k}(x)|^2dx = \frac{\beta}{\alpha^2}\frac{1}{2}\int_0^{4K(k)}|\,\mathrm{sn}(y,k)|^2dy \\ &= \frac{4}{\beta|b|}(K(k)-E(k)) = \frac{16}{|b|T}K(k)(K(k)-E(k)), \end{aligned}$$

which is a strictly increasing function of $k\in(0,1)$ with range $(0,\infty)$ and hence has an inverse function. ☐

3.4. **Orbital Stability.** Recall that we say that a standing wave $\psi(t,x)=e^{-iat}u(x)$ is orbitally stable for the flow of (1.1) in the function space X if for all $\varepsilon>0$ there exists $\delta>0$ such that the following holds: if $\psi_0\in X$ verifies

$$\|\psi_0-u\|_X \leqslant \delta$$

then the solution ψ of (1.1) with initial data $\psi(0,x)=\psi_0$ verifies for all $t\in\mathbb{R}$ the estimate

$$\inf_{\theta\in\mathbb{R},y\in\mathbb{R}}\|\psi(t,\cdot)-e^{i\theta}u(\cdot-y)\|_X<\varepsilon.$$

As an immediate corollary of the variational characterizations above, we have the following orbital stability statements:

Corollary 3.9. *The standing wave $\psi(t,x) = e^{-iat}u(x)$ is a solution of (1.1), and is orbitally stable in X in the following cases. For Jacobi elliptic functions: for any $k \in (0,1)$,*

$$\begin{aligned}
&a = 1 + k^2, && b = -2k^2, && u = \mathrm{sn}(\cdot,k), && X = H^1_{\mathrm{loc}} \cap A^-_{2K};\\
&a = 1 - 2k^2, && b = 2k^2, && u = \mathrm{cn}(\cdot,k), && X = H^1_{\mathrm{loc}} \cap A_{2K};\\
&a = -(2 - k^2), && b = 2, && u = \mathrm{dn}(\cdot,k), && X = H^1_{\mathrm{loc}} \cap P_{2K}.
\end{aligned}$$

For constants and plane waves: $(b \neq 0)$

$$\begin{aligned}
&a = -\frac{2bm}{T}, && -\infty < b \leqslant \frac{\pi^2}{Tm}, && u = \sqrt{\frac{2m}{T}}, && X = H^1_{\mathrm{loc}} \cap P_T;\\
&a = \frac{4\pi^2}{T^2} - \frac{2bm}{T}, && b < 0, && u = e^{\pm\frac{2i\pi x}{T}}\sqrt{\frac{2m}{T}}, && X = H^1_{\mathrm{loc}} \cap A_{T/2}.
\end{aligned}$$

The proof of this corollary uses the variational characterizations from Propositions 3.2, 3.3, 3.4, 3.6, and 3.8. Note that for all the minimization problems considered we have the compactness of minimizing sequences. The proof follows the standard line introduced by Cazenave and Lions [8], we omit the details here.

Remark 3.10. The orbital stability of sn [13] in $H^1_{\mathrm{loc}} \cap A_{T/2}$ was proved using the Grillakis-Shatah-Strauss [18, 19] approach, which amounts to identifying the periodic wave as a *local* constrained minimizer in this subspace. So the above may be considered an alternate proof, using *global* variational information. In the case of sn, without Conjecture 3.7, some additional spectral information in the subspace $A^+_{T/2}$ is needed to obtain orbital stability in $H^1_{\mathrm{loc}} \cap A_{T/2}$ (rather than just $H^1_{\mathrm{loc}} \cap A^-_{T/2}$) – see Corollary 4.7 in the next section for this.

Orbital stability of cn was obtained in [13] only for small amplitude cn. We extend this result to all possible values of $k \in (0,1)$.

Remark 3.11. Using the complete integrability of (1.1), Bottman, Deconinck and Nivala [5], and Gallay and Pelinovsky [15] showed that sn is in fact a minimizer of a higher-order functional in $H^2_{\mathrm{loc}} \cap P_{nT}$ for any $n \in \mathbb{N}$, and thus showed it is orbitally stable in these spaces.

4. Spectral Stability

Given a standing wave $\psi(t,x) = e^{-iat}u(x)$ solution of (1.1), we consider the linearization of (1.1) around this solution: if $\psi(t,x) = e^{-iat}(u(x) + h)$, then h verifies

$$i\partial_t h - Lh + N(h) = 0,$$

where L denotes the linear part and N the nonlinear part. Assuming u is real-valued, we separate h into real and imaginary parts to get the equation

$$\partial_t \begin{pmatrix} \mathcal{Re}(h) \\ \mathcal{Im}(h) \end{pmatrix} = J\mathcal{L}\begin{pmatrix} \mathcal{Re}(h) \\ \mathcal{Im}(h) \end{pmatrix} + \begin{pmatrix} -\mathcal{Im}(N(h)) \\ \mathcal{Re}(N(h)) \end{pmatrix},$$

where

$$\mathcal{L} = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{aligned} L_+ &= -\partial_{xx} - a - 3b\,u^2, \\ L_- &= -\partial_{xx} - a - b\,u^2. \end{aligned}$$

We call

$$J\mathcal{L} = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix} \tag{4.1}$$

the *linearized operator* of (1.1) about the standing wave $e^{-iat}u(x)$.

Now suppose $u \in H^1_{\text{loc}} \cap P_T$ is a (period T) periodic wave, and consider its linearized operator $J\mathcal{L}$ as an operator on the Hilbert space $(P_T)^2$, with domain $(H^2_{\text{loc}} \cap P_T)^2$. The main structural properties of $J\mathcal{L}$ are:

- since $L_\pm$ are self-adjoint operators on P_T, $\mathcal{L}$ is self-adjoint on $(P_T)^2$, while J is skew-adjoint and unitary
$$\mathcal{L}^* = \mathcal{L}, \quad J^* = -J = J^{-1}, \tag{4.2}$$
- $J\mathcal{L}$ commutes with complex conjugation,
$$\overline{J\mathcal{L}\, f} = J\mathcal{L}\bar{f}, \tag{4.3}$$
- $J\mathcal{L}$ is antisymmetric under conjugation by the matrix
$$C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
(which corresponds to the operation of complex conjugation *before* complexification),
$$J\mathcal{L}C = -CJ\mathcal{L}. \tag{4.4}$$

At the *linear* level, the stability of the periodic wave is determined by the location of the spectrum $\sigma(J\mathcal{L})$, which in this periodic setting consists of isolated eigenvalues of finite multiplicity [29]. We first make the standard observation that as a result of (4.3) and (4.4), the spectrum of $J\mathcal{L}$ is invariant under reflection about the real and imaginary axes:

$$\lambda \in \sigma(J\mathcal{L}) \implies \pm\lambda,\ \pm\bar{\lambda} \in \sigma(J\mathcal{L}).$$

Indeed, if $J\mathcal{L}f = \lambda f$, then

$$(4.3) \implies J\mathcal{L}\bar{f} = \bar{\lambda}\bar{f}, \quad (4.4) \implies J\mathcal{L}Cf = -\lambda Cf,$$
$$(4.3) \text{ and } (4.4) \implies J\mathcal{L}C\bar{f} = -\bar{\lambda}C\bar{f}.$$

We are interested in whether the entire spectrum of $J\mathcal{L}$ lies on the imaginary axis, denoted $\sigma(J\mathcal{L}|_{P_T}) \subset i\mathbb{R}$, in which case we say the periodic wave u is *spectrally stable in* P_T. Moreover, if $S \subset P_T$ is an invariant subspace – more precisely, $J\mathcal{L} : (H^2_{\text{loc}} \cap S)^2 \to (S)^2$ – then we will say that the periodic wave u is *spectrally stable in* S if the entire $(S)^2$ spectrum of $J\mathcal{L}$ lies on the imaginary axis, denoted $\sigma(J\mathcal{L}|_S) \subset i\mathbb{R}$. In particular, for $k \in (0,1)$ and $K = K(k)$, since sn^2, cn^2, dn^2 $\in P^+_{2K}$, the corresponding linearized operators respect the decomposition (2.2), and we may consider $\sigma(J\mathcal{L}|_S)$ for $S = P^\pm_{2K}, A^\pm_{2K} \subset P_{4K}$, with

$$\begin{aligned} \sigma\left(J\mathcal{L}|_{P_{4K}}\right) &= \sigma\left(J\mathcal{L}|_{P_{2K}}\right) \cup \sigma\left(J\mathcal{L}|_{A_{2K}}\right) \\ &= \sigma\left(J\mathcal{L}|_{P^+_{2K}}\right) \cup \sigma\left(J\mathcal{L}|_{P^-_{2K}}\right) \cup \sigma\left(J\mathcal{L}|_{A^+_{2K}}\right) \cup \sigma\left(J\mathcal{L}|_{A^-_{2K}}\right). \end{aligned} \tag{4.5}$$

Of course, spectral stability (which is purely linear) is a weaker notion than orbital stability (which is nonlinear). Indeed, the latter implies the former – see Proposition 4.10 and the remarks preceding it.

The main result of this section is the following.

Theorem 4.1. *Spectral stability in P_T, $T = 4K(k)$, holds for:*

- $u = \mathrm{sn}$, $k \in (0,1)$,
- $u = \mathrm{cn}$ *and* $k \in (0, k_c)$, *where* k_c *is the unique* $k \in (0,1)$ *so that* $K(k) = 2E(k)$, $k_c \approx 0.908$.

Remark 4.2. The function $f(k) = K(k) - 2E(k)$ is strictly increasing in $k \in (0,1)$, (since $K(k)$ is increasing while $E(k)$ is decreasing in k), with $f(0) = -\frac{\pi}{2}$ and $f(1) = \infty$.

Remark 4.3. Using Evans function techniques, it was proved in [21] that $\sigma(J\mathcal{L}^{\mathrm{cn}}) \subset i\mathbb{R}$ also for $k \in [k_c, 1)$. This fact is also supported by numerical evidence (see Section 7).

Remark 4.4. In the case of sn, the $H^2_{\mathrm{loc}} \cap P_{nT}$ orbital stability obtained in [5, 15] (using integrability) immediately implies spectral stability in P_{nT}, and in particular in P_T. So our result for sn could be considered an alternate, elementary proof, not relying on the integrability.

Remark 4.5. The spectral stability of dn in P_{2K} (its own fundamental period) is an immediate consequence of its orbital stability in $H^1_{\mathrm{loc}} \cap P_{2K}$, see Proposition 4.10.

4.1. **Spectra of L_+ and L_-.** We assume now that we are given $k \in (0,1)$ and we describe the spectrum of L_+ and L_- in P_{4K} when ϕ is cn, dn or sn. When $\phi = \mathrm{sn}$, we denote L_+ by L_+^{sn}, and we use similar notations for L_- and cn, dn. Due to the algebraic relationships between cn, dn and sn, we have

$$\begin{aligned}
L_+^{\mathrm{sn}} &= -\partial_{xx} - (1+k^2) + 6k^2 \,\mathrm{sn}^2, \\
L_+^{\mathrm{cn}} &= -\partial_{xx} - (1-2k^2) - 6k^2 \,\mathrm{cn}^2 = L_+^{\mathrm{sn}} - 3k^2, \\
L_+^{\mathrm{dn}} &= -\partial_{xx} + (2-k^2) - 6 \,\mathrm{dn}^2 = L_+^{\mathrm{sn}} - 3.
\end{aligned}$$

Similarly for L_-, we obtain

$$\begin{aligned}
L_-^{\mathrm{sn}} &= -\partial_{xx} - (1+k^2) + 2k^2 \,\mathrm{sn}^2, \\
L_-^{\mathrm{cn}} &= -\partial_{xx} - (1-2k^2) - 2k^2 \,\mathrm{cn}^2 = L_-^{\mathrm{sn}} + k^2, \\
L_-^{\mathrm{dn}} &= -\partial_{xx} + (2-k^2) - 2 \,\mathrm{dn}^2 = L_-^{\mathrm{sn}} + 1.
\end{aligned}$$

As a consequence, $L_\pm^{\mathrm{sn}}$, $L_\pm^{\mathrm{cn}}$, and $L_\pm^{\mathrm{dn}}$ share the same eigenvectors. Moreover, these operators enter in the framework of Schrödinger operators with periodic potentials and much can be said about their spectrum (see e.g. [10, 29]). Recall in particular that given a Schrödinger operator $L = -\partial_{xx} + V$ with periodic potential V of period T, the eigenvalues λ_n of L on P_T satisfy

$$\lambda_0 < \lambda_1 \leqslant \lambda_2 < \lambda_3 \leqslant \lambda_4 < \cdots,$$

with corresponding eigenfunctions ψ_n such that ψ_0 has no zeros, ψ_{2m+1} and ψ_{2m+2} have exactly $2m+2$ zeros in $[0,T)$ ([10, p. 39]). From the equations satisfied by cn, dn, sn, we directly infer that

$$L_-^{\mathrm{sn}} \,\mathrm{dn} = -\,\mathrm{dn}, \qquad L_-^{\mathrm{sn}} \,\mathrm{cn} = -k^2 \,\mathrm{cn}, \qquad L_-^{\mathrm{sn}} \,\mathrm{sn} = 0.$$

Taking the derivative with respect to x of the equations satisfied by cn, dn, sn, we obtain

$$L_+^{\mathrm{sn}} \partial_x \,\mathrm{sn} = 0, \qquad L_+^{\mathrm{sn}} \partial_x \,\mathrm{cn} = 3k^2 \partial_x \,\mathrm{cn}, \qquad L_+^{\mathrm{sn}} \partial_x \,\mathrm{dn} = 3 \partial_x \,\mathrm{dn}\,.$$

Looking for eigenfunctions in the form $\chi = 1 - A\,\mathrm{sn}^2$ for $A \in \mathbb{R}$, we find two other eigenfunctions:

$$L_+^{\mathrm{sn}}\chi_- = e_-\chi_-, \qquad\qquad L_+^{\mathrm{sn}}\chi_+ = e_+\chi_+,$$

where

$$\chi_\pm = 1 - \left(k^2 + 1 \pm \sqrt{k^4 - k^2 + 1}\right)\mathrm{sn}^2,$$
$$\pm e_\pm = \pm\left(k^2 + 1 \pm 2\sqrt{k^4 - k^2 + 1}\right) > 0.$$

In the interval $[0, 4K)$, χ_- has no zero, sn_x and cn_x have two zeros each, while dn_x and χ_+ have 4 zeros each. By Sturm-Liouville theory, they are the first 5 eigenvectors of L_+ for each of sn, cn, and dn, and all other eigenfunctions have strictly greater eigenvalues. Similarly, $\mathrm{dn} > 0$ has no zeros, while cn and sn have two each, so these are the first 3 eigenfunctions of L_- for each of sn, cn, and dn, and all other eigenfunctions have strictly greater eigenvalues.

The spectra of $L_\pm^{\mathrm{sn}}$, $L_\pm^{\mathrm{cn}}$, and $L_\pm^{\mathrm{dn}}$ are represented in Figure 4.1, where the eigenfunctions are also classified with respect to the subspaces of decomposition (2.2).

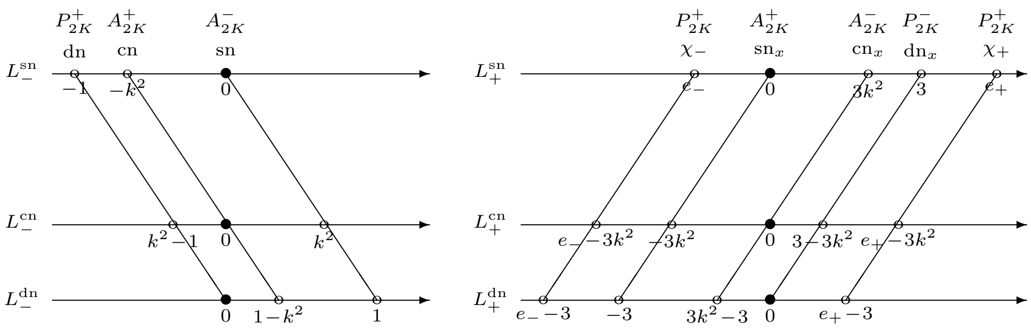

FIGURE 4.1. Eigenvalues for L_- and L_+ in P_{4K}.

We may now recover the result of [13] that sn is orbitally stable in $H^1_{\mathrm{loc}} \cap A_{2K}$, using the following simple consequences of the spectral information above:

Lemma 4.6. *There exists $\delta > 0$ such that the following coercivity properties hold.*

(1) $L_+^{\mathrm{sn}}|_{A_{2K}^-} > \delta$,
(2) $L_-^{\mathrm{sn}}|_{A_{2K}^- \cap \{\mathrm{sn}\}^\perp} > \delta$,
(3) $L_+^{\mathrm{sn}}|_{A_{2K}^+ \cap \{(\mathrm{sn})_x\}^\perp} > \delta$,
(4) $L_-^{\mathrm{sn}}|_{A_{2K}^+ \cap \{(\mathrm{sn})_x\}^\perp} > \delta$.

Proof. The first three are immediate from figure 4.1 (note the first two also follow from the minimization property Proposition 3.8), while we see that in A_{2K}^+, $L_+^{\mathrm{sn}}|_{\{(\mathrm{sn})_x\}^\perp} > e_+$, so since $\mathrm{sn}^2(x) \leqslant 1$,

$$L_-^{\mathrm{sn}}|_{\{(\mathrm{sn})_x\}^\perp} = \left(L_+^{\mathrm{sn}} - 4k^2\,\mathrm{sn}^2\right)|_{\{(\mathrm{sn})_x\}^\perp} > e_+ - 4k^2 > 0$$

where the last inequality is easily verified. □

Corollary 4.7. *For all $k \in (0,1)$, the standing wave $\psi(t,x) = e^{-i(1+k^2)t}\,\mathrm{sn}(x,k)$ is orbitally stable in $H^1_{\mathrm{loc}} \cap A_{2K}$.*

Proof. Lemma 4.6 shows that sn is a non-degenerate (up to phase and translation) *local* minimizer of the energy with fixed mass and momentum. So the classical Cazenave-Lions [8] argument yields the orbital stability. □

Finally, we also record here the following computations concerning $L_{\pm}^{\mathrm{cn}}$, used in analyzing the generalized kernel of $J\mathcal{L}^{\mathrm{cn}}$ in the next subsection:

Lemma 4.8. *Define $\hat{E}(x,k) = E(\phi,k)|_{\sin\phi = \mathrm{sn}(x,k)}$. Let ϕ_1 and ξ_1 be given by the following expressions.*

$$\phi_1 = \frac{\left(\hat{E}(x,k) - \frac{E}{K}x\right)\mathrm{cn}_x - k^2\,\mathrm{cn}^3 + \frac{Kk^2-E}{K}\,\mathrm{cn}}{2(2k^2-1)\frac{E}{K} + 2(1-k^2)},$$

$$\xi_1 = \frac{\left(\hat{E}(x,k) - \frac{E}{K}x\right)\mathrm{cn} + \mathrm{cn}_x}{-2(1-k^2) + \frac{2E}{K}}.$$

The denominators are positive and we have

$$L_+^{\mathrm{cn}}\phi_1 = \mathrm{cn}, \qquad L_-^{\mathrm{cn}}\xi_1 = \mathrm{cn}_x\,.$$

Note $\hat{E}$ and ξ_1 are odd while ϕ_1 is even. In particular $(\phi_1, \mathrm{cn}_x) = 0 = (\xi_1, \mathrm{cn})$. Moreover, $L_+^{\mathrm{cn}}(\frac{1}{2}\mathrm{cn} - (1-2k^2)\phi_1) = \mathrm{cn}_{xx}$.

Proof. Recall that the elliptic integral of the second kind $\hat{E}(x,k)$ is not periodic. In fact, it is asymptotically linear in x and verifies

$$\hat{E}(x + 2K, k) = \hat{E}(x,k) + 2E(k).$$

By (2.10), $\partial_x \hat{E}(x,k) = \mathrm{dn}^2(x,k)$. Denote $L_\pm = L_\pm^{\mathrm{cn}}$ in this proof. Using (2.4) and (2.5), we have

$$\begin{aligned}
L_+\,\mathrm{cn} &= -4k^2\,\mathrm{cn}^3,\\
L_+(x\,\mathrm{cn}_x) &= 4k^2\,\mathrm{cn}^3 - 2(2k^2-1)\,\mathrm{cn},\\
L_+\,\mathrm{cn}^3 &= 6k^2\,\mathrm{cn}^5 - 8(2k^2-1)\,\mathrm{cn}^3 - 6(1-k^2)\,\mathrm{cn},\\
L_+(\hat{E}(x,k)\,\mathrm{cn}_x) &= 6k^4\,\mathrm{cn}^5 - 4k^2(3k^2-2)\,\mathrm{cn}^3 + 2(1-4k^2+3k^4)\,\mathrm{cn}\,.
\end{aligned}$$

Define

$$\tilde{\phi}_1 = \left(\hat{E}(x,k) - \frac{E(k)}{K(k)}x\right)\mathrm{cn}_x - k^2\,\mathrm{cn}^3 + \frac{K(k)k^2 - E(k)}{K(k)}\,\mathrm{cn}\,.$$

Then $\tilde{\phi}_1$ is periodic (of period $4K$) and verifies

$$L_+\tilde{\phi}_1 = \left(2(2k^2-1)\frac{E(k)}{K(k)} + 2(1-k^2)\right)\mathrm{cn}\,.$$

The factor is positive if $2k^2 \geq 1$. If $2k^2 < 1$, it is greater than $2(2k^2-1)+2(1-k^2) = 2k^2$. Define,

$$\phi_1 = \left(2(2k^2-1)\frac{E(k)}{K(k)} + 2(1-k^2)\right)^{-1}\tilde{\phi}_1.$$

Then

$$L_+\phi_1 = \mathrm{cn}\,.$$

As for L_-, we have

$$\begin{aligned} L_-(\mathrm{cn}_x) &= 4k^2\,\mathrm{cn}^2\,\mathrm{cn}_x,\\ L_-(x\,\mathrm{cn}) &= -2\,\mathrm{cn}_x,\\ L_-(\hat{E}(x,k)\,\mathrm{cn}) &= -2(1-k^2)\,\mathrm{cn}_x - 4k^2\,\mathrm{cn}^2\,\mathrm{cn}_x\,. \end{aligned}$$

Define

$$\tilde{\xi}_1 = \left(\hat{E}(x,k) - \frac{E(k)}{K(k)}x\right)\mathrm{cn} + \mathrm{cn}_x\,.$$

Then $\tilde{\xi}_1$ is periodic (of period $4K$) and verifies

$$L_-\tilde{\xi}_1 = \left(-2(1-k^2) + \frac{2E(k)}{K(k)}\right)\mathrm{cn}_x\,.$$

The factor is positive by (2.11). Defining

$$\xi_1 = \left(-2(1-k^2) + \frac{2E(k)}{K(k)}\right)^{-1}\tilde{\xi}_1$$

we get $L_-\xi_1 = \mathrm{cn}_x$. The last statement of the lemma follows from (2.8). $\square$

4.2. **Orthogonality Properties.** The following lemma records some standard properties of eigenvalues and eigenfunctions of the linearized operator $J\mathcal{L}$, which follow only from the structural properties (4.2) and (4.4):

Lemma 4.9. *The following properties hold.*

(1) *(symplectic orthogonality of eigenfunctions) Let $f = (f_1, f_2)^T$ and $g = (g_1, g_2)^T$ be two eigenvectors of $J\mathcal{L}$ corresponding to eigenvalues $\lambda, \mu \in \mathbb{C}$. Then* (4.2) *implies*

$$\lambda + \bar{\mu} \neq 0 \implies (f, Jg) = (f, \mathcal{L}g) = 0,$$

while (4.4) *implies*

$$\lambda - \bar{\mu} \neq 0 \implies (Cf, Jg) = (Cf, \mathcal{L}g) = 0,$$

so that

$$\lambda \pm \bar{\mu} \neq 0 \implies (f_1, g_2) = (f_2, g_1) = 0.$$

(2) *(unstable eigenvalues have zero energy) If $J\mathcal{L}f = \lambda f$, $\lambda \notin i\mathbb{R}$, then* (4.2) *implies*

$$(f, \mathcal{L}f) = 0.$$

Proof. We first prove (1). We have

$$\lambda\,(f, Jg) = (\lambda f, Jg) = (J\mathcal{L}f, Jg) = (\mathcal{L}f, g) = (f, \mathcal{L}g) = -\,(f, \mu Jg) = -\bar{\mu}\,(f, Jg)\,,$$

so $(\lambda + \bar{\mu})\,(f, Jg) = 0$ which gives the first statement. The second statement follows from the same argument with f replaced by Cf, while the third statement is a consequence of $(f, Jg) = (Cf, Jg) = 0$.

Item (2) is a special case of the first statement of (1), with $g = f$. $\square$

4.3. **Spectral Stability of** sn **and** cn**.** Our goal in this section is to establish Theorem 4.1, i.e. to prove the spectral stability of sn in P_{4K} for all $k \in (0,1)$, and the spectral stability of cn in P_{4K} for all $k \in (0, k_c)$.

We first recall the standard fact that

$$\text{orbital stability} \implies \text{spectral stability.}$$

Indeed, an eigenvalue $\lambda = \alpha + i\beta$ of $J\mathcal{L}$ with $\alpha > 0$ produces a solution of the linearized equation whose magnitude grows at the exponential rate $e^{\alpha t}$, and this linear growing mode (together with its orthogonality properties from Lemma 4.9) can be used to contradict orbital stability. Rather than go through the nonlinear dynamics, however, we will give a simple direct proof of spectral stability in the symmetry subspaces where we have the orbital stability – that is, in P_{2K} for dn, and in A_{2K} for cn and sn – using just the spectral consequences for $L_\pm$ implied by the (local) minimization properties of these elliptic functions:

Proposition 4.10. *For $0 < k < 1$, $K = K(k)$,* dn *is spectrally stable in P_{2K} while* cn *and* sn *are spectrally stable in A_{2K}. Precisely, we have*

$$\sigma(J\mathcal{L}^{\mathrm{dn}}|_{P_{2K}}) \subset i\mathbb{R}, \quad \sigma(J\mathcal{L}^{\mathrm{cn}}|_{A_{2K}}) \subset i\mathbb{R}, \quad \sigma(J\mathcal{L}^{\mathrm{sn}}|_{A_{2K}}) \subset i\mathbb{R}.$$

Proof. Begin with dn in P_{2K}. From Figure 4.1, we see $L_-^{\mathrm{dn}}|_{\mathrm{dn}^\perp} > 0$, and thus $(L_-^{\mathrm{dn}})^{\pm 1/2}$ exist on $\mathrm{dn}^\perp$. It follows from the minimization property Proposition 3.2 that on $\mathrm{dn}^\perp$, $L_+^{\mathrm{dn}} \geqslant 0$ (otherwise there is a perturbation of dn lowering the energy while preserving the mass). Suppose $J\mathcal{L}^{\mathrm{dn}} f = \lambda f$, $\lambda \notin i\mathbb{R}$. Then $L_-^{\mathrm{dn}} L_+^{\mathrm{dn}} f_1 = -\lambda^2 f_1$. Since $(\mathrm{dn}, 0)^T$ is an eigenvector of $J\mathcal{L}$ for the eigenvalue 0, Lemma 4.9 implies $f_1 \perp \mathrm{dn}$. Therefore, we have

$$(L_-^{\mathrm{dn}})^{1/2} L_+^{\mathrm{dn}} (L_-^{\mathrm{dn}})^{1/2} \left((L_-^{\mathrm{dn}})^{-1/2} f_1 \right) = -\lambda^2 \left((L_-^{\mathrm{dn}})^{-1/2} f_1 \right)$$

and on $\mathrm{dn}^\perp$,

$$L_+ \geqslant 0 \implies L_-^{1/2} L_+ L_-^{1/2} \geqslant 0 \implies \lambda^2 \leqslant 0$$

contradicting $\lambda \notin i\mathbb{R}$.

Next, consider cn in A_{2K}. Again from Figure 4.1, we see $L_-^{\mathrm{cn}}|_{\mathrm{cn}^\perp} > 0$, while the minimization property Proposition 3.4 implies that $L_+^{\mathrm{cn}} \geqslant 0$ on $\mathrm{cn}^\perp$, and so the spectral stability follows just as for dn above.

Finally, consider sn in A_{2K}. By Lemma 4.6, $L_+^{\mathrm{sn}} > 0$ on $\{(\mathrm{sn})_x\}^\perp$, while $L_-^{\mathrm{sn}} \geqslant 0$ on $\{(\mathrm{sn})_x\}^\perp$, and so the spectral stability follows from the same argument as above, with the roles of L_+ and L_- reversed. □

Moreover, both sn and cn are spectrally stable in P_{2K}^-:

Lemma 4.11. *For $0 < k < 1$, $K = K(k)$,*

$$\sigma(J\mathcal{L}^{\mathrm{cn}}|_{P_{2K}^-}) \subset i\mathbb{R}, \quad \sigma(J\mathcal{L}^{\mathrm{sn}}|_{P_{2K}^-}) \subset i\mathbb{R}.$$

Proof. This is an immediate consequence of the positivity of $\mathcal{L}^{\mathrm{sn}}$ and $\mathcal{L}^{\mathrm{cn}}$ on P_{2K}^- (see Figure 4.1), and Lemma 4.9. □

So in light of (4.5), to prove Theorem 4.1, it remains only to show $\sigma(J\mathcal{L}|_{P_{2K}^+}) \subset i\mathbb{R}$ for each of cn and sn.

This will follow from a simplified version of a general result for infinite dimensional Hamiltonian systems (see [20, 22, 23]) relating coercivity of the linearized energy with the number of eigenvalues with negative Krein signature of the linearized operator $J\mathcal{L}$ of the form (4.1):

Lemma 4.12 (coercivity lemma)**.** *Consider $J\mathcal{L}$ on $S \times S$ for some invariant subspace $S \subset P_T$, and suppose it has an eigenvalue whose eigenfunction $\xi = (\xi_1, \xi_2)^T$ has negative (linearized) energy:*

$$J\mathcal{L}\xi = \mu\xi, \quad (\xi, \mathcal{L}\xi) < 0.$$

Then the following results hold.

(1) *If L_+ has a one-dimensional negative subspace (in S):*

$$L_+ f = -\lambda f, \quad \lambda > 0, \quad L_+|_{f^\perp} > 0 \tag{4.6}$$

Then $L_+|_{\xi_2^\perp} > 0$.

(2) *If L_- has a one-dimensional negative subspace (in S):*

$$L_- g = -\nu g, \quad \nu > 0, \quad L_-|_{g^\perp} > 0 \tag{4.7}$$

Then $L_-|_{\xi_1^\perp} > 0$.

(3) *If both (4.6) and (4.7) hold, then $\sigma(J\mathcal{L}|_{S\times S}) \subset i\mathbb{R}$.*

Proof. First note that by Lemma 4.9 (2), $0 \neq \mu \in i\mathbb{R}$, and writing $\mu = i\gamma$, $0 \neq \gamma \in \mathbb{R}$, we have $L_-\xi_2 = i\gamma\xi_1$, $L_+\xi_1 = -i\gamma\xi_2$.

Moreover,

$$(\xi_1, L_+\xi_1) = -\gamma\,(\xi_1, i\xi_2) = \gamma\,(i\xi_1, \xi_2) = (L_-\xi_2, \xi_2)\,,$$

so by assumption $(\xi_1, L_+\xi_1) = (\xi_2, L_-\xi_2) < 0$.

We prove (1). For any $h \perp \xi_2$, decompose

$$h = \alpha f + h_+, \quad \xi_1 = \beta f + \xi_+, \quad h_+ \perp f, \quad \xi_+ \perp f,$$

where we may assume $\alpha \geq 0$ and $\beta \geq 0$. We have

$$0 = i\gamma\,(h, \xi_2) = (h, -i\gamma\xi_2) = (h, L_+\xi_1) = -\lambda\alpha\beta + (h_+, L_+\xi_+)\,.$$

Thus, using $L_+|_{f^\perp} > 0$, $L_+^{1/2} = (L_+|_{f^\perp})^{1/2}$ is well defined on $f^\perp$ and

$$\begin{aligned}(\alpha\beta\lambda)^2 = (h_+, L_+\xi_+)^2 &= \left(L_+^{1/2}h_+, L_+^{1/2}\xi_+\right)^2 \\ &\leq (h_+, L_+h_+)\,(\xi_+, L_+\xi_+) \\ &= ((h, L_+h) + \alpha^2\lambda)((\xi_1, L_+\xi_1) + \beta^2\lambda)\end{aligned}$$

with both factors on the right > 0. Since $(\xi_1, L_+\xi_1) < 0$, we must have $(h, L_+h) > 0$.

Statement (2) follows in exactly the same way, with the roles of L_+ and L_- reversed, the roles of ξ_1 and ξ_2 reversed, and with g and ν replacing f and λ.

Finally, for (3), suppose $J\mathcal{L}\eta = \zeta\eta$. If $\zeta \notin i\mathbb{R}$, then by Lemma 4.9 (1), $(\xi_1, \eta_2) = (\xi_2, \eta_1) = 0$, and so by parts (1) and (2),

$$(\eta_1, L_+\eta_1) > 0, \quad (\eta_2, L_+\eta_2) > 0, \quad \implies (\eta, \mathcal{L}\eta) > 0,$$

contradicting Lemma 4.9 (2). Thus $\zeta \in i\mathbb{R}$. □

Proof of Theorem 4.1. Begin with sn in P_{2K}^+. From Figure 4.1 it is clear that in P_{2K}^+, condition (4.6) holds for L_+^{sn} and (4.7) holds for L_-^{sn}. Explicit computation yields

$$L_+^{\mathrm{sn}}(\mathrm{dn}^2 + k^2\,\mathrm{cn}^2) = -(1-k^2)^2, \quad L_-^{\mathrm{sn}}1 = -(\mathrm{dn}^2 + k^2\,\mathrm{cn}^2),$$

which implies

$$J\mathcal{L}^{\mathrm{sn}}\begin{pmatrix}\mathrm{dn}^2+k^2\,\mathrm{cn}^2\\ i(1-k^2)\end{pmatrix}=i(1-k^2)\begin{pmatrix}\mathrm{dn}^2+k^2\,\mathrm{cn}^2\\ i(1-k^2)\end{pmatrix}.$$

Moreover,

$$\begin{aligned}\left\langle\mathcal{L}^{\mathrm{sn}}\begin{pmatrix}\mathrm{dn}^2+k^2\,\mathrm{cn}^2\\ i(1-k^2)\end{pmatrix},\begin{pmatrix}\mathrm{dn}^2+k^2\,\mathrm{cn}^2\\ i(1-k^2)\end{pmatrix}\right\rangle&\\ =\left\langle L_+^{\mathrm{sn}}(\mathrm{dn}^2+k^2\,\mathrm{cn}^2),\mathrm{dn}^2+k^2\,\mathrm{cn}^2\right\rangle+(1-k^2)\left\langle L_-^{\mathrm{sn}}1,1\right\rangle&\\ =-((1-k^2)^2+(1-k^2))\left\langle 1,(\mathrm{dn}^2+k^2\,\mathrm{cn}^2)\right\rangle&\\ =-((1-k^2)^2+(1-k^2))(4E(k)-2(1-k^2)K(k))<0,&\end{aligned}$$

by (2.11). Hence all the conditions of Lemma 4.12 are verified for sn in P_{2K}^+, and so we conclude $\sigma(J\mathcal{L}^{\mathrm{sn}}|_{P_{2K}^+})\subset i\mathbb{R}$, as required.

Next we turn to cn. Again from Figure 4.1 it is clear that in P_{2K}^+, condition (4.6) holds for L_+^{cn} and (4.7) holds for L_-^{cn}. Explicit computation yields

$$L_+^{\mathrm{cn}}(-\mathrm{dn}^2+k^2\,\mathrm{sn}^2)=1,\quad L_-^{\mathrm{cn}}1=-\mathrm{dn}^2+k^2\,\mathrm{sn}^2,$$

which implies

$$J\mathcal{L}^{\mathrm{cn}}\begin{pmatrix}-\mathrm{dn}^2+k^2\,\mathrm{sn}^2\\ i\end{pmatrix}=i\begin{pmatrix}-\mathrm{dn}^2+k^2\,\mathrm{sn}^2\\ i\end{pmatrix}.$$

Moreover, when $k<k_c$, we have

$$\left\langle\mathcal{L}^{\mathrm{cn}}\begin{pmatrix}-\mathrm{dn}^2+k^2\,\mathrm{sn}^2\\ i\end{pmatrix},\begin{pmatrix}-\mathrm{dn}^2+k^2\,\mathrm{sn}^2\\ i\end{pmatrix}\right\rangle=2\left\langle L_-^{\mathrm{sn}}1,1\right\rangle=4K(k)-8E(k)<0.$$

Hence the conditions of Lemma 4.12 are verified for cn in P_{2K}^+ when $k<k_c$, yielding $\sigma(J\mathcal{L}^{\mathrm{cn}}|_{P_{2K}^+})\subset i\mathbb{R}$, as required. □

5. Linear Instability

Theorem 4.1 (and Proposition 4.10) give the spectral stability of the periodic waves dn, sn, and cn (at least for $k<k_c$) against perturbations which are periodic with their fundamental period. It is also natural to ask if this stability is maintained against perturbations whose period is a *multiple* of the fundamental period. In light of Bloch-Floquet theory, this question is also relevant for stability against *localized* perturbations in $L^2(\mathbb{R})$.

5.1. **Theoretical Analysis.** It is a simple observation that dn immediately becomes unstable against perturbations with twice its fundamental period:

Proposition 5.1. *Both* $\sigma(J\mathcal{L}^{\mathrm{dn}}|_{A_{2K}^+})$ *and* $\sigma(J\mathcal{L}^{\mathrm{dn}}|_{A_{2K}^-})$ *contain a pair of non-zero real eigenvalues. In particular* dn *is linearly unstable against perturbations in* P_{4K}*.*

Proof. In each of A_{2K}^+ and A_{2K}^-, $L_-^{\mathrm{dn}}>0$ while L_+^{dn} has a negative eigenvalue: $L_+^{\mathrm{dn}}f=-\lambda f$, $\lambda>0$. So the self-adjoint operator $(L_-^{\mathrm{dn}})^{1/2}L_+^{\mathrm{dn}}(L_-^{\mathrm{dn}})^{1/2}$ has a negative direction,

$$\Big((L_-^{\mathrm{dn}})^{-1/2}f,\ ((L_-^{\mathrm{dn}})^{1/2}L_+^{\mathrm{dn}}(L_-^{\mathrm{dn}})^{1/2})(L_-^{\mathrm{dn}})^{-1/2}f\Big)=-\lambda(f,\ f)<0,$$

hence a negative eigenvalue $(L_-^{\mathrm{dn}})^{1/2} L_+^{\mathrm{dn}} (L_-^{\mathrm{dn}})^{1/2} g = -\mu^2 g$, $\mu > 0$. Setting $h := (L_-^{\mathrm{dn}})^{-1/2} g$, $h \in A_{2K}^+$ (A_{2K}^-), we see

$$L_+^{\mathrm{dn}} L_-^{\mathrm{dn}} h = -\mu^2 h \quad \Longrightarrow \quad J\mathcal{L}^{\mathrm{dn}} \begin{pmatrix} L_-^{\mathrm{dn}} h \\ \pm \mu h \end{pmatrix} = \pm \mu \begin{pmatrix} L_-^{\mathrm{dn}} h \\ \pm \mu h \end{pmatrix}.$$

Hence $\mu, -\mu \in \mathbb{R}$ are eigenvalues of $J\mathcal{L}$ in A_{2K}^+ (A_{2K}^-). $\square$

Remark 5.2. The proof shows dn is unstable in P_{2nK} for every even n since $h \in P_{2nK}$. In fact, dn is unstable in any P_{2nK}, $n \geq 2$. Indeed, we always have $L_- \,\mathrm{dn} = 0$, thus by Sturm-Liouville Theory (see e.g. [10, Theorem 3.1.2]), 0 is always the first simple eigenvalue of L_- in P_{2nK}. Moreover, $L_+ \,\mathrm{dn}_x = 0$, and dn_x has $2n$ zeros in P_{2nK}. Hence there are at least $2n-2$ negative eigenvalues for L_+ in P_{2nK}. With the above argument, this proves linear instability in P_{2nK} for any $n \geq 2$.

For sn, the $H^2(\mathbb{R})$ orbital stability result of [5, 15] implies spectral stability against perturbations which are periodic with *any* multiple of the fundamental period.

Using formal perturbation theory, [30] showed that cn becomes unstable against perturbations which are periodic with period a sufficiently large multiple of the fundamental period. Our main goal in this section is to make this rigorous:

Theorem 5.3. *For $0 < k < 1$, there exists $n_1 = n_1(k) \in \mathbb{N}$ such that* cn *is linearly unstable in P_{4nK} for $n \geq n_1$, i.e., the spectrum of $J\mathcal{L}^{\mathrm{cn}}$ as an operator on P_{4nK} contains an eigenvalue with positive real part.*

We will in fact prove a slightly more general result, which is the existence of a branch strictly contained in the first quadrant for the spectrum of $J\mathcal{L}^{\mathrm{cn}}$ considered as an operator on $L^2(\mathbb{R})$. Theorem 5.3 will be a consequence of a more general perturbation result applying to all real periodic waves (see Proposition 5.4), and in particular not relying on any integrable structure.

We start with some preliminaries. Let

$$J\mathcal{L} = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}$$

with

$$L_- = -\partial_{xx} - a - bu^2, \quad L_+ = -\partial_{xx} - a - 3bu^2$$

where u a periodic solution to

$$u_{xx} + au + b|u|^2\, u = 0. \tag{5.1}$$

We assume that $u(x) \in \mathbb{R}$ and let T denote a period of u^2. The spectrum of $J\mathcal{L}$ as an operator on $L^2(\mathbb{R})$ can be analyzed using Bloch-Floquet decomposition. For $\theta \in [0, 2\pi/T)$, define

$$J\mathcal{L}^\theta = \begin{pmatrix} 0 & L_-^\theta \\ -L_+^\theta & 0 \end{pmatrix}$$

where $L_\pm^\theta$ is the operator obtained when formally replacing ∂_x by $\partial_x + i\theta = e^{-i\theta x}\partial_x \left(e^{i\theta x}\cdot\right)$ in the expression of $L_\pm$. If we let $(M^\theta f)(x) = e^{i\theta x} f(x)$, then $L_\pm^\theta = M^{-\theta} L_\pm M^\theta$. Then we have

$$\sigma\left(J\mathcal{L}|_{L^2(\mathbb{R})}\right) = \bigcup_{\theta \in [0, \frac{2\pi}{T}]} \sigma\left(J\mathcal{L}^\theta|_{P_T}\right). \tag{5.2}$$

In what follows, all operators are considered on P_T unless otherwise mentioned.

Let us consider the case $\theta = \frac{\pi}{T}$. Denote

$$D = \partial_x + i\frac{\pi}{T}.$$

Since u is a real valued periodic solution to (5.1), by Lemmas 2.1 and 2.2, u is a rescaled cn, dn or sn. In any case, the following holds:

$$\begin{gathered}\varphi = e^{-i\frac{\pi}{T}x}u, \;\; \psi = D\varphi = e^{-i\frac{\pi}{T}x}u_x \;\in H^1_{\mathrm{loc}} \cap P_T \setminus \{0\} \\ \text{are such that } \ker(L_-^{\frac{\pi}{T}}) = \langle\varphi\rangle, \;\; \ker(L_+^{\frac{\pi}{T}}) = \langle\psi\rangle.\end{gathered} \tag{5.3}$$

Note that for any $f, g \in H^1_{\mathrm{loc}} \cap P_T$, we can integrate by parts with D:

$$\int_0^T Df\bar{g}\,dx = -\int_0^T f\overline{Dg}\,dx.$$

Remark that

$$(\varphi,\psi) = \int_0^T \varphi\bar{\psi}\,dx = \int_0^T uu_x\,dx = 0.$$

Therefore there exist φ_1, ψ_1 such that

$$L_-^{\frac{\pi}{T}}\psi_1 = \psi, \quad L_+^{\frac{\pi}{T}}\varphi_1 = \varphi, \quad \varphi_1 \perp \psi, \quad \psi_1 \perp \varphi.$$

The kernel of the operator $J\mathcal{L}^{\frac{\pi}{T}}$ is generated by $\binom{\psi}{0}$, $\binom{0}{\varphi}$. On top of that, the generalized kernel of $J\mathcal{L}^{\frac{\pi}{T}}$ contains (at least) $\binom{0}{\psi_1}$, $\binom{\varphi_1}{0}$.

Our goal is to analyze the spectrum of the operator $J\mathcal{L}^{\frac{\pi}{T}-\varepsilon}$ when $|\varepsilon|$ is small. In particular, we want to locate the eigenvalues generated by perturbation of the generalized kernel of $J\mathcal{L}^{\frac{\pi}{T}}$. For the sake of simplicity in notation, when $\theta = \frac{\pi}{T}$, we use a tilde to replace the exponent $\frac{\pi}{T}$. In particular, we write

$$J\mathcal{L}^{\frac{\pi}{T}} = J\tilde{\mathcal{L}}, \quad L_\pm^{\frac{\pi}{T}} = \tilde{L}_\pm.$$

Proposition 5.4. *Assume the condition (5.16) stated below. There exist $\lambda_1 \in \mathbb{C}$ with $\mathcal{Re}(\lambda_1) > 0$, $\mathcal{Im}(\lambda_1) > 0$; $b_0 \in \mathbb{C}$; and $\varepsilon_0 > 0$, such that for all $0 \leqslant \varepsilon < \varepsilon_0$ there exist $\lambda_2(\varepsilon) \in \mathbb{C}$, $b_1(\varepsilon) \in \mathbb{C}$, $v_2(\varepsilon), w_2(\varepsilon) \in H^2_{\mathrm{loc}} \cap P_T$,*

$$\begin{gathered}|b_1(\varepsilon)| + |\lambda_2(\varepsilon)| + \|v_2(\varepsilon)\|_{H^2_{\mathrm{loc}}\cap P_T} + \|w_2(\varepsilon)\|_{H^2_{\mathrm{loc}}\cap P_T} \lesssim 1 \\ v_2(\varepsilon) \perp \psi, \quad w_2(\varepsilon) \perp \varphi,\end{gathered} \tag{5.4}$$

verifying the following property. Set

$$v_0 = b_0\psi, \qquad v_1 = b_1(\varepsilon)\psi - 2ib_0\tilde{L}_+^{-1}D\psi - \lambda_1\varphi_1, \tag{5.5}$$

$$w_0 = \varphi, \qquad w_1 = (b_0\lambda_1 - 2i)\psi_1. \tag{5.6}$$

Here, $\tilde{L}_+^{-1}$ is taken from $\psi^\perp$ to $\psi^\perp$. Define

$$\begin{array}{lclclcl} \lambda & = & & & \varepsilon\lambda_1 & + & \varepsilon^2\lambda_2(\varepsilon), \\ v & = & v_0 & + & \varepsilon v_1(\varepsilon) & + & \varepsilon^2 v_2(\varepsilon), \\ w & = & w_0 & + & \varepsilon w_1 & + & \varepsilon^2 w_2(\varepsilon). \end{array}$$

Then

$$J\mathcal{L}^{\frac{\pi}{T}-\varepsilon}\begin{pmatrix} v \\ w \end{pmatrix} = \lambda\begin{pmatrix} v \\ w \end{pmatrix}.$$

Note that the orthogonality conditions in (5.4) are reasonable: The eigenvector is normalized by $P_\varphi w = w_0 = \varphi$, and hence $w_2 \perp \varphi$. To impose $v_2 \perp \psi$, we allow $b_1(\varepsilon)\psi$ in v_1 to be ε-dependent to absorb $P_\psi(v - v_0)$.

Proof of Proposition 5.4. Let us write the expansion of the operators in ε. We have

$$L_\pm^{\frac{\pi}{T}-\varepsilon} = L_\pm^{\frac{\pi}{T}} + 2i\varepsilon D + \varepsilon^2,$$

Therefore

$$J\mathcal{L}^{\frac{\pi}{T}-\varepsilon} = J\mathcal{L}^{\frac{\pi}{T}} + \varepsilon \begin{pmatrix} 0 & 2iD \\ -2iD & 0 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

We expand in ε the equation $\left(J\mathcal{L}^{\frac{\pi}{T}-\varepsilon} - \lambda\mathcal{I}\right)\binom{v}{w} = 0$ and show that it can be satisfied at each order of ε.

At order $\mathcal{O}(1)$, we have

$$J\tilde{\mathcal{L}}\begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = 0,$$

which is satisfied because $\binom{v_0}{w_0} \in \ker(J\tilde{\mathcal{L}})$ by definition.

At order $\mathcal{O}(\varepsilon)$, we have

$$J\tilde{\mathcal{L}}\begin{pmatrix} v_1 \\ w_1 \end{pmatrix} + \begin{pmatrix} -\lambda_1 & 2iD \\ -2iD & -\lambda_1 \end{pmatrix}\begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = 0,$$

which can be rewritten, using the expression of v_0, w_0, and $D\varphi = \psi$, as

$$\tilde{L}_- w_1 = (b_0\lambda_1 - 2i)\psi, \tag{5.7}$$

$$\tilde{L}_+ v_1 = -2ib_0 D\psi - \lambda_1\varphi. \tag{5.8}$$

It is clear that the functions $v_1(\varepsilon)$ and w_1 defined in (5.5)-(5.6) satisfy (5.7)-(5.8).

At order $\mathcal{O}(\varepsilon^2)$, we consider the equation as a whole, involving also the higher orders of ε. We have

$$\begin{aligned} J\tilde{\mathcal{L}}\begin{pmatrix} v_2 \\ w_2 \end{pmatrix} + \begin{pmatrix} -\lambda_1 & 2iD \\ -2iD & -\lambda_1 \end{pmatrix}\begin{pmatrix} v_1 \\ w_1 \end{pmatrix} + \begin{pmatrix} -\lambda_2 & 1 \\ -1 & -\lambda_2 \end{pmatrix}\begin{pmatrix} v_0 \\ w_0 \end{pmatrix} & \\ + \varepsilon\left(\begin{pmatrix} -\lambda_1 & 2iD \\ -2iD & -\lambda_1 \end{pmatrix}\begin{pmatrix} v_2 \\ w_2 \end{pmatrix} + \begin{pmatrix} -\lambda_2 & 1 \\ -1 & -\lambda_2 \end{pmatrix}\begin{pmatrix} v_1 \\ w_1 \end{pmatrix}\right) & \\ + \varepsilon^2\begin{pmatrix} -\lambda_2 & 1 \\ -1 & -\lambda_2 \end{pmatrix}\begin{pmatrix} v_2 \\ w_2 \end{pmatrix} &= 0, \end{aligned}$$

in other words

$$\tilde{L}_- w_2 = W_2 + \varepsilon W_3 + \varepsilon^2 W_4 \tag{5.9}$$

$$\tilde{L}_+ v_2 = V_2 + \varepsilon V_3 + \varepsilon^2 V_4 \tag{5.10}$$

where

$$\begin{aligned} &W_2 = \lambda_1 v_1 - 2iDw_1 + \lambda_2 v_0 - w_0, && V_2 = -2iDv_1 - \lambda_1 w_1 - v_0 - \lambda_2 w_0, \\ &W_3 = \lambda_1 v_2 - 2iDw_2 + \lambda_2 v_1 - w_1, && V_3 = -2iDv_2 - \lambda_1 w_2 - v_1 - \lambda_2 w_1, \\ &W_4 = \lambda_2 v_2 - w_2, && V_4 = -v_2 - \lambda_2 w_2. \end{aligned} \tag{5.11}$$

Note that V_2 and W_2 depend on b_0, λ_1 and b_1, λ_2, whereas V_3, V_4 and W_3, W_4 also depend on v_2 and w_2. Our strategy to solve the system (5.9)-(5.10) is divided into two steps. We first ensure that it can be solved at the main order by ensuring that the compatibility conditions

$$(W_2, \varphi) = (V_2, \psi) = 0 \tag{5.12}$$

are satisfied. This is achieved by making a suitable choice of b_0, λ_1. Then we solve for b_1, λ_2, v_2, w_2 by using a Lyapunov-Schmidt argument.

We rewrite the compatibility conditions (5.12) in the following form, using the expressions for v_0, w_0, v_1 and w_1, and the properties of φ and ψ:

$$\begin{aligned}(\varphi_1,\varphi)\,\lambda_1^2 + b_0 2i\left((D\psi,\varphi_1)-(\psi_1,\psi)\right)\lambda_1 + \quad & \left((\varphi,\varphi)-4\,(\psi_1,\psi)\right)=0\\ b_0\,(\psi_1,\psi)\,\lambda_1^2 + \quad 2i\left((\varphi_1,D\psi)-(\psi_1,\psi)\right)\lambda_1 + b_0 & \left((\psi,\psi)-4\left(\tilde{L}_+^{-1}D\psi,D\psi\right)\right)=0\end{aligned}$$

These equations do not depend on b_1 or λ_2 although W_2 and V_2 do. For a moment, we write these equations as

$$A_1\lambda_1^2 + b_0 B\lambda_1 + \quad C_1 = 0, \tag{5.13}$$

$$b_0 A_2\lambda_1^2 + \quad B\lambda_1 + b_0 C_2 = 0, \tag{5.14}$$

where

$$\begin{aligned}A_1 &:= (\varphi_1,\varphi) \quad \in\mathbb{R},\\ A_2 &:= (\psi_1,\psi) \quad \in\mathbb{R},\\ B &:= 2i\left((D\psi,\varphi_1)-(\psi_1,\psi)\right) \quad \in i\mathbb{R},\\ C_1 &:= (\varphi,\varphi)-4\,(\psi_1,\psi) \quad \in\mathbb{R},\\ C_2 &:= (\psi,\psi)-4\left(\tilde{L}_+^{-1}D\psi,D\psi\right) \quad \in\mathbb{R}.\end{aligned}$$

Multiplying (5.13) by $C_2 + A_2\lambda_1^2$, (5.14) by $B\lambda_1$, and subtracting gives

$$A_1A_2\lambda_1^4 + (A_1C_2 + A_2C_1 - B^2)\lambda_1^2 + C_1C_2 = 0, \tag{5.15}$$

a quadratic equation in λ_1^2 with real coefficients. If $A_1A_2 \neq 0$, the roots of (5.15) are given by

$$\lambda_1^2 = \frac{-(A_1C_2 + A_2C_1 - B^2) \pm \sqrt{(A_1C_2 + A_2C_1 - B^2)^2 - 4A_1A_2C_1C_2}}{2A_1A_2}$$

We now assume the discriminant of this quadratic is negative:

$$(A_1C_2 + A_2C_1 - B^2)^2 - 4A_1A_2C_1C_2 < 0 \tag{5.16}$$

which implies that $A_1A_2 \neq 0$, and moreover guarantees the existence of a root λ_1 of (5.15) strictly contained in the first quadrant: $\mathcal{R}e\lambda_1 > 0$ and $\mathcal{I}m\lambda_1 > 0$ (the other roots being $-\lambda_1$, $\pm\bar{\lambda}_1$). It follows from (5.16) that $B \neq 0$, and so we may solve (5.13) and set

$$b_0 := -\frac{(A_1\lambda_1^2 + C_1)}{B\lambda_1},$$

so that both (5.13) and (5.14) are satisfied.

We now solve for b_1, λ_2, v_2, w_2 using a Lyapunov-Schmidt argument. The first step is to solve, given (b_1,λ_2), projected versions of (5.9)-(5.10),

$$\begin{aligned}\tilde{L}_- w_2 &= W_2 + P_{\varphi^\perp}\left[\varepsilon W_3 + \varepsilon^2 W_4\right]\\ \tilde{L}_+ v_2 &= V_2 + P_{\psi^\perp}\left[\varepsilon V_3 + \varepsilon^2 V_4\right]\end{aligned} \tag{5.17}$$

to obtain $v_2 = v_2(b_1,\lambda_2) \in \psi^\perp$, $w_2 = w_2(b_1,\lambda_2) \in \varphi^\perp$:

Lemma 5.5. *Given any $b_1 \in \mathbb{C}$, $\lambda_2 \in \mathbb{C}$ with $|b_1| + |\lambda_2| \leqslant M$, there is a unique solution*

$$(v_2,w_2) = (v_2(b_1,\lambda_2), w_2(b_1,\lambda_2)) \in (H^2_{loc}\cap P_T\cap\psi^\perp)\times(H^2_{loc}\cap P_T\cap\varphi^\perp)$$

of (5.17), with $\|v_2\|_{H^2} + \|w_2\|_{H^2} \leqslant C(M)$.

Proof. By the expressions (5.11), we may rewrite system (5.17) as a linear system of v_2 and w_2,

$$\tilde{\mathcal{L}}_\varepsilon \begin{pmatrix} v_2 \\ w_2 \end{pmatrix} = \begin{pmatrix} S_v \\ S_w \end{pmatrix} + \varepsilon \begin{pmatrix} R_v \\ R_w \end{pmatrix},$$

where

$$\tilde{\mathcal{L}}_\varepsilon = \begin{pmatrix} \tilde{L}_+ + P_{\psi^\perp} 2i\varepsilon D + \varepsilon^2 & (\varepsilon\lambda_1 + \varepsilon^2\lambda_2) P_{\psi^\perp} \\ -(\varepsilon\lambda_1 + \varepsilon^2\lambda_2) P_{\varphi^\perp} & \tilde{L}_- + P_{\varphi^\perp} 2i\varepsilon D + \varepsilon^2 \end{pmatrix},$$

$$\begin{aligned} S_v &= P_{\psi^\perp}(-2iD[b_1\psi - 2ib_0\tilde{L}_+^{-1}D\psi - \lambda_1\varphi_1] - \lambda_1(b_0\lambda_1 - 2i)\psi_1 - \lambda_2\varphi) \\ S_w &= P_{\varphi^\perp}(\lambda_1[b_1\psi - 2ib_0\tilde{L}_+^{-1}D\psi - \lambda_1\varphi_1] - 2iD(b_0\lambda_1 - 2i)\psi_1 + \lambda_2 b_0\psi) \\ R_v &= P_{\psi^\perp}(\lambda_2(2i - b_0\lambda_1)\psi_1 - [-2ib_0\tilde{L}_+^{-1}D\psi - \lambda_1\varphi_1]) \\ R_w &= P_{\varphi^\perp}(\lambda_2[b_1\psi - 2ib_0\tilde{L}_+^{-1}D\psi - \lambda_1\varphi_1] + (2i - b_0\lambda_1)\psi_1). \end{aligned} \tag{5.18}$$

Note that S_v, S_w, R_v and R_w do not contain v_2, w_2 or ε. Recalling the definition

$$\tilde{\mathcal{L}} = \begin{pmatrix} \tilde{L}_+ & 0 \\ 0 & \tilde{L}_- \end{pmatrix},$$

it follows from (5.3) that

$$\tilde{\mathcal{L}}^{-1} : (P_T \cap \psi^\perp) \times (P_T \cap \varphi^\perp) \to (P_T \cap H^2_{loc} \cap \psi^\perp) \times (P_T \cap H^2_{loc} \cap \varphi^\perp)$$

is bounded, and hence so is $\tilde{\mathcal{L}}_\varepsilon^{-1}$, uniformly in ε for ε sufficiently small, with

$$\|\tilde{\mathcal{L}}_\varepsilon^{-1} - \tilde{\mathcal{L}}^{-1}\|_{(L^2\times L^2 \to H^2\times H^2)} \lesssim \varepsilon.$$

Thus

$$\begin{pmatrix} v_2 \\ w_2 \end{pmatrix} = \tilde{\mathcal{L}}_\varepsilon^{-1}\left(\begin{pmatrix} S_v \\ S_w \end{pmatrix} + \varepsilon \begin{pmatrix} R_v \\ R_w \end{pmatrix} \right) = \begin{pmatrix} \tilde{L}_+^{-1} S_v \\ \tilde{L}_-^{-1} S_w \end{pmatrix} + O_{H^2\times H^2}(\varepsilon) \tag{5.19}$$

gives $(v_2(b_1, \lambda_2), w_2(b_1, \lambda_2))$ as desired. □

The second step is to plug $(v_2(b_1, \lambda_2), w_2(b_1, \lambda_2))$ back into V_3, V_4, W_3, W_4, and solve, for (b_1, λ_2), the remaining compatibility conditions

$$(V_3 + \varepsilon V_4, \psi) = (W_3 + \varepsilon W_4, \varphi) = 0 \tag{5.20}$$

which, together with (5.17), complete the solution of the eigenvalue problem. Using (5.19) and (5.11), we may write (5.20) as the system

$$\begin{aligned} 0 &= (-2iD[\tilde{L}_+^{-1} S_v + O(\varepsilon)] - \lambda_1[\tilde{L}_-^{-1} S_w + O(\varepsilon)] - v_1 - \lambda_2 w_1 + \varepsilon V_4, \psi) \\ 0 &= (\lambda_1[\tilde{L}_+^{-1} S_v + O(\varepsilon)] - 2iD[\tilde{L}_-^{-1} S_w + O(\varepsilon)] + \lambda_2 v_1 - w_1 + \varepsilon W_4, \varphi) \end{aligned}$$

and then by the expressions (5.5)-(5.6) and (5.18), we may further rewrite as

$$\Phi(b_1, \lambda_2, \varepsilon) = (M + O(\varepsilon)) \begin{pmatrix} b_1 \\ \lambda_2 \end{pmatrix} + F + O(\varepsilon) = 0 \tag{5.21}$$

where Φ is a rational vector function of b_1, λ_2 and ε; F is a fixed (independent of (b_1,λ_2)) vector with $|F| \lesssim 1$; and $M = \frac{\partial \Phi}{\partial(b_1,\lambda_2)}|_{\varepsilon=0}$ is the matrix

$$M = \begin{pmatrix} (-4D\tilde{L}_+^{-1}D\psi - \lambda_1^2\psi_1 - \psi, \psi) & (2iD\varphi_1 - 2(\lambda_1 b_0 - i)\psi_1, \psi) \\ (-2i\lambda_1\tilde{L}_+^{-1}D\psi - 2i\lambda_1 D\psi_1, \varphi) & (-\lambda_1\varphi_1 - 2ib_0 D\psi_1 - 2ib_0\tilde{L}_+^{-1}D\psi - \lambda_1\varphi_1, \varphi) \end{pmatrix}$$
$$= \begin{pmatrix} 4(\tilde{L}_+^{-1}D\psi, D\psi) - \lambda_1^2(\psi_1,\psi) - (\psi,\psi) & -2i(\varphi_1, D\psi) - 2(\lambda_1 b_0 - i)(\psi_1,\psi) \\ -2i\lambda_1(D\psi,\varphi_1) + 2i\lambda_1(\psi_1,\psi) & -2\lambda_1(\varphi_1,\varphi) + 2ib_0(\psi_1,\psi) - 2ib_0(D\psi,\varphi_1) \end{pmatrix}$$
$$= \begin{pmatrix} -C_2 - A_2\lambda_1^2 & -B - 2\lambda_1 b_0 A_2 \\ -\lambda_1 B & -b_0 B - 2\lambda_1 A_1 \end{pmatrix} = \begin{pmatrix} \frac{B}{b_0}\lambda_1 & -B - 2\lambda_1 b_0 A_2 \\ -\lambda_1 B & -b_0 B - 2\lambda_1 A_1 \end{pmatrix}.$$

where in the last step we used (5.14). The determinant of M is, using (5.14) and (5.13) to eliminate b_0,

$$\begin{aligned} \det M &= -2\lambda_1 B\left(B + \frac{\lambda_1}{b_0}A_1 + \lambda_1 b_0 A_2\right) \\ &= -2\lambda_1 B\left(B - \frac{A_2\lambda_1^2 + C_2}{B}A_1 - \frac{A_1\lambda_1^2 + C_1}{B}A_2\right) \\ &= -2\lambda_1\left(B^2 - 2A_1A_2\lambda_1^2 - C_2A_1 - C_1A_2\right). \end{aligned}$$

Since A_1, A_2, C_1, C_2, B^2 are real, and $\lambda_1 A_1 A_2 \neq 0$, we have $\det M \neq 0$, otherwise $\lambda_1^2 \in \mathbb{R}$.

Thus (b_1,λ_2) may be solved from (5.21) for ε sufficiently small by the implicit function theorem, providing the required solution to (5.20), and so completing the proof of Proposition 5.4. □

Proof of Theorem 5.3. We need only verify the assumptions of Proposition 5.4 for the case of $u(x) = \mathrm{cn}(x;k)$, $T = 2K(k)$. Since $u = \mathrm{cn} \in A_{2K}$, we have $u^2 = \mathrm{cn}^2 \in P_T$. Moreover, (5.3) holds (see Figure 4.1). It remains to verify the condition (5.16). The values of the coefficients for the equations of b_0 and λ_1 are given by the following formulas, obtained by using the equation verified by cn and the explicit expressions given by Lemma 4.8. Due to the complicated nature of the expressions, the dependence of E and K on k will be left implicit.

$$\begin{aligned} A_1 &= (\varphi_1,\varphi) = (\phi_1,u) = \frac{k^2K(2E-K) + (E-K)^2}{2k^2(E(1-2k^2) - K(1-k^2))}, \\ A_2 &= (\psi_1,\psi) = (\xi_1,u_x) = \frac{k^2K(2E-K) + (E-K)^2}{2k^2(E-(1-k^2)K)}, \\ B &= 2i\left((\varphi_1,D\psi) - (\psi_1,\psi)\right) = 2i\left((\phi_1,u_{xx}) - A_2\right) \\ &= -i\frac{2EK(k-1)(k+1)(K-E)}{(E(1-2k^2) - K(1-k^2))(E-(1-k^2)K)}, \\ C_1 &= (\varphi,\varphi) - 4(\psi_1,\psi) = (u,u) - 4A_2 = \frac{2K^2(k-1)(k+1)}{E-(1-k^2)K}, \\ C_2 &= (\psi,\psi) - 4\left\langle \tilde{L}_+^{-1}D\psi, D\psi\right\rangle = (u_x,u_x) - 4\left\langle L_+^{-1}u_{xx}, u_{xx}\right\rangle \\ &= \frac{2K^2(k-1)(k+1)}{E(1-2k^2) - K(1-k^2)}. \end{aligned}$$

Therefore,

$$(A_1C_2 + A_2C_1 - B^2)^2 - 4A_1A_2C_1C_2 = -\frac{16K^4E^2(1-k)^3(1+k)^3(K-E)^2}{k^2(E-(1-k^2)K)^2(E(1-2k^2)-(1-k^2)K)^2} < 0,$$

Thus Proposition (5.4) applies, providing an unstable eigenvalue of $J(\mathcal{L}^{\mathrm{cn}})^\theta$ for $\theta = \frac{\pi}{2K} - \varepsilon$, and all $0 < \varepsilon \leqslant \varepsilon_0$. It follows in particular that cn is unstable against perturbations with period $4nK$, where n is the smallest even integer $\geqslant \frac{\pi}{K\varepsilon_0}$. This concludes the proof of Theorem 5.3. □

5.2. **Numerical Spectra.** We have tested numerically the spectra of the different operators involved. To this aim, we used a fourth order centered finite difference discretization of the second derivative operator. Unless otherwise specified, we have used 2^{10} grid points. The spectra are then obtained using the built in function of our scientific computing software (Scilab). Whenever the spectra can be theoretically described, the theoretical description and our numerical computations are in good agreement.

We start by the presentation of the spectra of $J\mathcal{L}^{\mathrm{pq}}$, for $\mathrm{pq} = \mathrm{cn}, \mathrm{dn}, \mathrm{sn}$ on P_{4K}.

Observation 5.6. *On P_{4K}, the spectrum of $J\mathcal{L}^{\mathrm{pq}}$ is such that*

- *if* $\mathrm{pq} = \mathrm{sn}$ *then* $\sigma(J\mathcal{L}^{\mathrm{sn}}) \subset i\mathbb{R}$ *for all* $k \in (0,1)$*,*
- *if* $\mathrm{pq} = \mathrm{cn}$*, then* $\sigma(J\mathcal{L}^{\mathrm{cn}}) \subset i\mathbb{R}$ *for all* $k \in (0,1)$*, including when* $k > k_c$*,*
- *if* $\mathrm{pq} = \mathrm{dn}$*, then* $J\mathcal{L}^{\mathrm{dn}}$ *admits two double eigenvalues* $\pm\lambda$ *with* $\lambda > 0$ *and the rest of the spectrum verifies* $(\sigma(J\mathcal{L}^{\mathrm{dn}}) \setminus \{\pm\lambda\}) \subset i\mathbb{R}$ *for all* $k \in (0,1)$*.*

The numerical observations for cn *and* dn *at* $k = 0.95$ *are represented in Figure 5.1.*

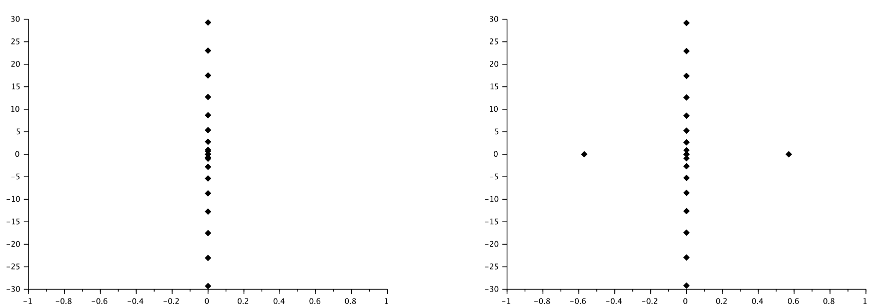

FIGURE 5.1. $\sigma(J\mathcal{L}^{\mathrm{cn}})$ (left) and $\sigma(J\mathcal{L}^{\mathrm{dn}})$ (right) on P_{4K} for $k = 0.95$

We then compare the results of Theorem 5.3 with the numerical results. In Figure 5.2, we have drawn the numerical spectrum of $J\mathcal{L}^{\mathrm{cn}}$ as an operator on $L^2(\mathbb{R})$. To this aim, we have used the Bloch decomposition of the spectrum of $J\mathcal{L}^{\mathrm{cn}}$ given in (5.2): we computed the spectrum of $J(\mathcal{L}^{\mathrm{cn}})^\theta$ for θ in a discretization of $(0, \frac{\pi}{2K}]$ and we have interpolated between the values obtained to get the curve in plain (blue) line. In order to keep the computation time reasonable, we have dropped the number of space points from 2^{10} to 2^8. We then have drawn in dashed (red) the straight lines passing through the origin and the points whose coordinates are given in the complex plane by $\pm\lambda_1, \pm\bar{\lambda}_1$, λ_1 given in the proof of Proposition 5.4. The picture shows that the dashed (red) line are tangent to the plain (blue) curve, thus confirming λ_1 as the first order in the expansion for the eigenvalue emerging from 0 performed in Proposition 5.4.

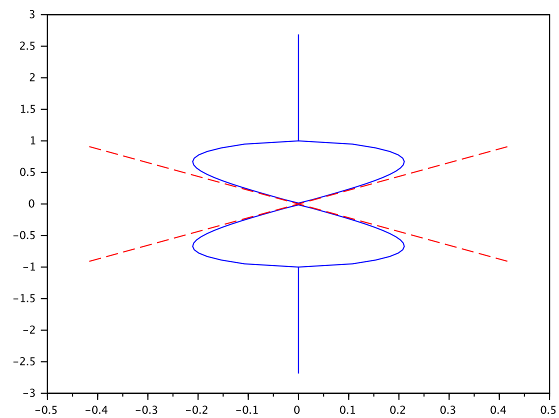

Figure 5.2. $\sigma(J\mathcal{L}^{\mathrm{cn}})$ on $L^2(\mathbb{R})$ for $k = 0.9$ (plain (blue) curve), first order asymptotic around 0 (dashed (red) lines)

Numerically, eigenvalues on the number 8 curve in Figure 5.2 are simple, and move from the origin toward the intersection points of the number 8 curve with the imaginary axis, when θ is decreased from $\pi/(2K)$ to 0^+.

These eigenvalues are simple because we did the Block decomposition (5.2) in P_{2K} with $\theta \in [0, 2\pi/T) = [0, \pi/K)$, and cn is only in $P_{2K}(-1)$, not in P_{2K}. Thus it is in the kernel of L_+^θ only for $\theta = \pi/(2K)$. The bifurcation occurs only near $\theta = \pi/(2K)$, not at $\theta = 0$.

In contrast, Rowlands [30] did the Block decomposition in P_{4K} with $\theta \in [0, \pi/(2K))$. We have cn $\in P_{4K}$, and cn is in the kernel of L_+^θ only for $\theta = 0$. The bifurcation occurs only near $\theta = 0$.

These two approaches are essentially the same, and our approach does not give a new instability branch.

6. Numerics

We describe here the numerical experiments performed to understand better the nature of the Jacobi elliptic functions as constrained minimizers of some functionals. To this aim, we use a normalized gradient flow approach related to the minimization problem (3.3).

6.1. **Gradient Flow With Discrete Normalization.** It is relatively natural when dealing with constrained minimization problems like (3.3)-(3.4) to use the following construction. Define an increasing sequence of time $0 = t_0 < \cdots < t_n$ and take an initial data u_0. Between each time step, let $u(t, x)$ evolve along the gradient flow

$$\begin{cases} u_t = -\mathcal{E}'(u) = u_{xx} + b|u|^2 u, \\ u(t_n, x) = u_n(x), \end{cases} \quad x \in \mathbb{R},\ t_n < t < t_{n+1},\ n \geqslant 0.$$

At each time step t_n, the function is renormalized so as to have the desired mass and momentum. The renormalization for the mass is obtained by a straightforward scaling:

$$u_{n+1}(x) := u(t_{n+1}, x)\sqrt{\frac{m}{\mathcal{M}(u(t_{n+1}, x))}}. \tag{6.1}$$

When there is no momentum, like in the minimization problems (3.1), (3.4), and only real-valued functions are considered, such approach to compute the minimizers was developed by Bao and Du [4].

However, dealing with complex valued solutions and with an additional momentum constraint as in problems (3.3), (3.5) turns out to make the problem more challenging and to our knowledge little is known about the strategies that one can use to deal with this situation (see [9] for an approach on a related problem).

To construct u_n in such a way that $\mathcal{P}(u_n) = p$, a simple scaling is not possible for at least two reasons. First of all, if $p = 0$, a scaling would obviously lead to failure of our strategy. Second, even if $p \neq 0$, as we are already using a scaling to get the correct mass, making a different scaling to obtain the momentum constraint will result into a modification of the mass. To overcome these difficulties, we propose the following approach.

Recall that, as noted in [4], the renormalizing step (6.1) is equivalent to solving exactly the following ordinary differential equation

$$u_t = \mu_n u, \quad t_n < t < t_{n+1}, \quad n \geqslant 0, \quad \mu_n = \frac{1}{t_{n+1} - t_n} \ln\left(\frac{\sqrt{2m}}{\|u(t_n)\|_{L^2}}\right). \tag{6.2}$$

Inspired by this remark, we consider the following problem, which we see as the equivalent of (6.2) for the momentum renormalization.

$$u_t = i\varpi_n u_x, \quad x \in \mathbb{R}, \quad t_n < t < t_{n+1}, \quad n \geqslant 0, \tag{6.3}$$

where we want to choose the values of ϖ_n in such a way that $\mathcal{P}(u(t_{n+1})) = p$. To this aim, we need to solve (6.3). Note that (6.3) is a partial differential equation, whereas (6.2) was only an ordinary differential equation. We make the following formal computations, which can be justified if the functions involved are regular enough. As we work with periodic functions, we consider the Fourier series representation of u, that is

$$u(t, x) = \sum_{j=-\infty}^{\infty} c_j(t) e^{i\frac{2\pi}{T}jx}$$

with the Fourier coefficients

$$c_j(t) = \frac{1}{T}\int_{-T/2}^{T/2} u(t, x) e^{-i\frac{2\pi}{T}jx} dx.$$

Then (6.3) becomes

$$\partial_t c_j = -\frac{2\pi}{T} j \varpi_n c_j, \quad j \in \mathbb{Z}, \quad t_n < t < t_{n+1}, \quad n \geqslant 0.$$

For each $j \in \mathbb{Z}$ and for any $t_n < t < t_{n+1}$ the solution is

$$c_j(t) = \exp\left(-\frac{2\pi}{T} j\varpi_n(t-t_n)\right) c_j(t_n),$$

and therefore the solution of (6.3) is

$$u(t,x) = \sum_{j=-\infty}^{\infty} \exp\left(-\frac{2\pi}{T} j\varpi_n(t-t_n)\right) c_j(t_n) e^{i\frac{2\pi}{T}jx}.$$

Using this Fourier series expansion of u, we have

$$\mathcal{P}(u(t_{n+1})) = -\sum_{j=-\infty}^{\infty} \pi j \exp\left(-\frac{4\pi}{T} j\varpi_n(t_{n+1}-t_n)\right) |c_j(t_n)|^2.$$

We determine implicitly the value of ϖ_n, by requiring that ϖ_n is such that

$$\mathcal{P}(u(t_{n+1})) = p.$$

In practice, it might not be so easy to compute ϖ_n and therefore we shall use the following approximation. We replace the exponential by its first order Maclaurin polynomial. We get

$$\mathcal{P}(u(t_{n+1})) = -\sum_{j=-\infty}^{\infty} \pi j \left(1 - \frac{4\pi}{T} j\varpi_n(t_{n+1}-t_n)\right) |c_j(t_n)|^2 + \mathcal{O}(\varpi_n^2(t_{n+1}-t_n)^2).$$

Therefore, an approximation for ϖ_n is given by $\tilde{\varpi}_n$, which is defined implicitly by

$$p = -\sum_{j=-\infty}^{\infty} \pi j \left(1 - \frac{4\pi}{T} j\tilde{\varpi}_n(t_{n+1}-t_n)\right) |c_j(t_n)|^2.$$

Solving for $\tilde{\varpi}_n$, we obtain

$$\tilde{\varpi}_n = \left(p + \sum_{j=-\infty}^{\infty} \pi j |c_j(t_n)|^2\right)\left((t_{n+1}-t_n)\frac{4\pi^2}{T}\sum_{j=-\infty}^{\infty} j^2 |c_j(t_n)|^2\right)^{-1}.$$

We can further simplify the expression of $\tilde{\varpi}_n$ by remarking that

$$\mathcal{P}(u(t_n)) = -\sum_{j=-\infty}^{\infty} \pi j |c_j(t_n)|^2, \quad \int_{-T/2}^{T/2} |\partial_x u(t_n)|^2 dx = \frac{4\pi^2}{T}\sum_{j=-\infty}^{\infty} j^2 |c_j(t_n)|^2.$$

This gives

$$\tilde{\varpi}_n = \frac{p - \mathcal{P}(u(t_n))}{(t_{n+1}-t_n)\|\partial_x u(t_n)\|_{L^2}^2}.$$

This is the value we will use in practice.

6.2. **Discretization.** Let us now further discretize our problem. We first present a semi-implicit time discretization, given by the following scheme.

$$\frac{\tilde{u}_{n+1} - u_n}{\delta t} = \partial_{xx}\tilde{u}_{n+1} + b|u_n|^2\tilde{u}_{n+1}, \quad \tilde{u}_{n+1} \in P_T,$$

$$\hat{u}_{n+1} = \sum_{j=-\infty}^{\infty} c_j(\tilde{u}_{n+1})\left(1 - \frac{2\pi}{T}\delta t\tilde{\varpi}_n j\right) e^{i\frac{2\pi}{T}jx},$$

$$u_{n+1} = \hat{u}_{n+1}\sqrt{\frac{m}{\mathcal{M}(\hat{u}_{n+1})}},$$

where $\tilde{\varpi}_n$ is given by

$$\tilde{\varpi}_n = \frac{p - \mathcal{P}(u_n)}{\delta t \|\partial_x u_n\|_{L^2}^2},$$

and $(c_j(\tilde{u}_{n+1}))$ are the Fourier coefficients of $\tilde{u}_{n+1}$. Note that the system is linear.

Remark 6.1. If $p = 0$, at the end of each step, u_{n+1} has the desired mass and momentum. If $p \neq 0$, then u_{n+1} only has the desired mass and it is unclear if the algorithm will still give convergence toward the desired mass-momentum constraint minimizer. We plan to investigate this question in further works.

Finally, we present the fully discretized problem. We discretize the space interval $\left[-\frac{T}{2}, \frac{T}{2}\right]$ by setting

$$x^0 = -\frac{T}{2}, \quad x^l = x^0 + l\delta x, \quad \delta x = \frac{T}{L}, \quad L \in 2\mathbb{N}.$$

We denote by u_n^l the numerical approximation of $u(t_n, x^l)$. Using the (backward Euler) semi-implicit scheme for time discretization and second-order centered finite difference for spatial derivatives, we obtain the following scheme.

$$\frac{\tilde{u}_{n+1}^l - u_n^l}{\delta t} = \frac{\tilde{u}_{n+1}^{l-1} - 2\tilde{u}_{n+1}^l + \tilde{u}_{n+1}^{l+1}}{\delta x^2} + b|u_n^l|^2 \tilde{u}_{n+1}^l, \quad u_{n+1}^0 = u_{n+1}^L, \tag{6.4}$$

$$\hat{u}_{n+1}^l = \sum_{j=-L/2}^{L/2} c_j(\tilde{u}_{n+1}) \left(1 - \frac{2\pi}{T}\delta t \tilde{\varpi}_n j\right) e^{i\frac{2\pi}{L} j l \delta x}, \tag{6.5}$$

$$\tilde{u}_{n+1}^l = \hat{u}_{n+1}^l \sqrt{\frac{m}{\mathcal{M}(\hat{u}_{n+1})}}, \tag{6.6}$$

where $c_j(\tilde{u}_{n+1}) = \frac{1}{L+1} \sum_{l=0}^{L} \tilde{u}_{n+1}^l e^{i\frac{2\pi}{L} j l \delta x}$.

As the system (6.4) is linear, we can solve it using a Thomas algorithm for tridiagonal matrix modified to take into account the periodic boundary conditions. The discrete Fourier transform and its inverse are computed using the built in Fast Fourier Transform algorithm.

We have not gone further in the analysis of the scheme presented above. As shown in the next section, the outcome of the numerical experiments are in good agreement with the theoretical results. We plan to further analyze and generalize our approach in future works.

7. Numerical Solutions of Minimization Problems

Before presenting the numerical experiments, we introduce some notation for particular plane waves. Define

$$\varphi_{\mu,\rho} = \sqrt{\frac{2\mu}{T}} e^{-i\frac{\rho}{\mu}x}, \text{ the plane wave with } \mathcal{M}(\varphi_{\mu,\rho}) = \mu \text{ and } \mathcal{P}(\varphi_{\mu,\rho}) = \rho.$$

In the numerical experiments, we have chosen to fix $k = 0.9$. The period will be either $T = 2K(k)$ or $T = 4K(k)$. We use 2^{10} grid points for the interval $[-\frac{T}{2}, \frac{T}{2}]$. The time step will be set to 1. We decided to run the algorithm until a maximal difference of 10^{-3} between the absolute values of the moduli of u_j^l and the expected minimizer has been reached.

We made the tests with the following initial data:

$$(a)\, u_0(x) = 5, \quad (b)\, u_0(x) = \exp(2i\pi x/T), \quad (c)\, u_0(x) = 1 + \cos(2i\pi x/T) + i. \tag{7.1}$$

Depending on the expected profile, we may have shifted u_j so that a minimum or a maximum of its modulus is at the boundary. Since the problem is translation invariant, this causes no loss of generality.

Since the initial data u_0 in (7.1) do not match the required mass/momentum, u_1 are very different from u_0. Thus (7.1) is a random choice, and this shows up in the rapid drop from t_0 to t_1 in Figure 7.1. The idea is to show that the choice of initial data is not important for the algorithm and that no matter from where the algorithm is starting, it converges to the supposed minimizer (unless the initial data has some symmetry preserved by the algorithm).

7.1. **Minimization Among Periodic Functions.** Minimization among periodic functions is completely covered by the theoretical results Propositions 3.2 and 3.3. We have performed different tests using the scheme described in (6.4)-(6.6) and we have found that the numerical results are in good agreement with the theoretical ones.

7.1.1. *The Focusing Case.* In all the experiments performed in this case, we have tested the scheme with and without the momentum renormalization step (6.6) and we have obtained the same result each time. This confirms that in the periodic case the momentum constraint plays no role (see (i) in Proposition 3.2, and Proposition 3.3). In what follows, we present only the results obtained using the full scheme with renormalization of mass and momentum.

We fix $T = 2K(k)$ and $b = 2$. We first perform an experiment to verify the agreement with case (ii) in Proposition 3.2. Let $m = \frac{\pi^2}{8K} < \frac{\pi^2}{bT}$. With each initial data in (7.1), we observe convergence towards the constant solution, hereby confirming case (ii) of Proposition 3.2. The results are presented in Figure 7.1 for initial data (c) of (7.1). The requested precision is achieved after 12 time steps.

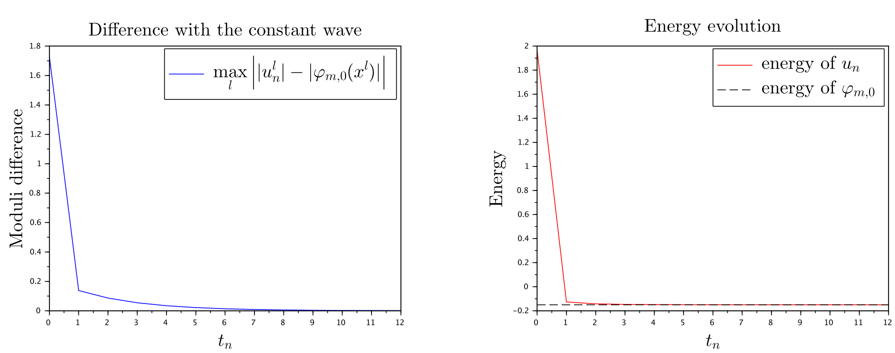

FIGURE 7.1. For $m = \frac{\pi^2}{8K} < \frac{\pi^2}{bT}$, focusing, periodic case

The second experiment that we perform is aimed at testing case (iv) of Proposition 3.2. Let $m = \mathcal{M}(\mathrm{dn}) = E(k)$. Once again we observe a good agreement between the theoretical prediction and the numerical experiment. The results are

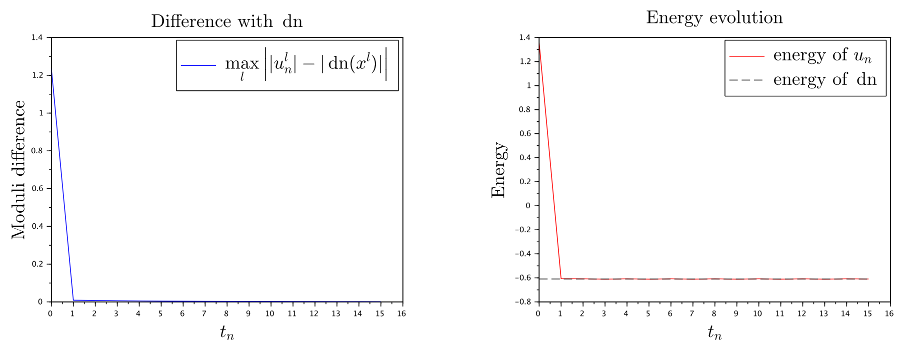

FIGURE 7.2. For $m = \mathcal{M}(\mathrm{dn}) = E(k)$, focusing, periodic case

presented in Figure 7.2 for initial data (c) of (7.1). The requested precision is achieved after 14 time steps.

All the other experiments that we have performed show a good agreement with the theoretical results in the focusing case for minimization among periodic functions. To avoid repetition, we give no further details here.

7.1.2. *The Defocusing Case.* We now present the experiment in the defocusing case. We have used $b = -2k^2$ and $T = 4K$. We have tested the algorithm with and without the momentum renormalization step (6.6), obtaining the same results. The results are presented in Figure 7.3 for initial data (c) of (7.1) and mass constraint $m = \mathcal{M}(\mathrm{sn}) = \frac{2(K-E)}{k^2}$. The requested precision is achieved after 6 time steps.

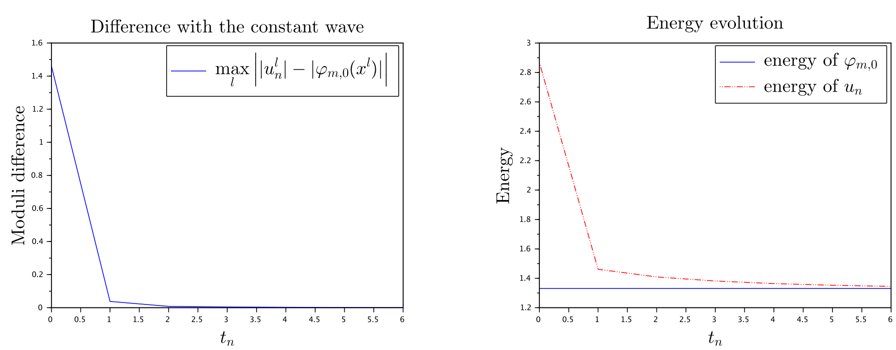

FIGURE 7.3. For $m = \mathcal{M}(\mathrm{sn}) = \frac{2(K-E)}{k^2}$, defocusing, periodic case

7.2. **Minimization Among Half-Anti-Periodic Functions.** We will in that case add an additional step in the algorithm in which we keep only the anti-periodic part of the function. This way it will not matter wether or not our initial data has the right anti-periodicity, since anti-periodicity will be forced at each iteration of the algorithm.

7.2.1. *The Focusing Case.* We compare in this section the numerical results with Proposition 3.4. We have used $b = 2k^2$ and $T = 4K$. The tests performed show a good agreement between the numerics and the theoretical result. We present in Figure 7.4 the result for initial data (c) of (7.1) and mass constraint $m = \mathcal{M}(\mathrm{cn}) = 2(E - (1-k^2)K)/k^2$

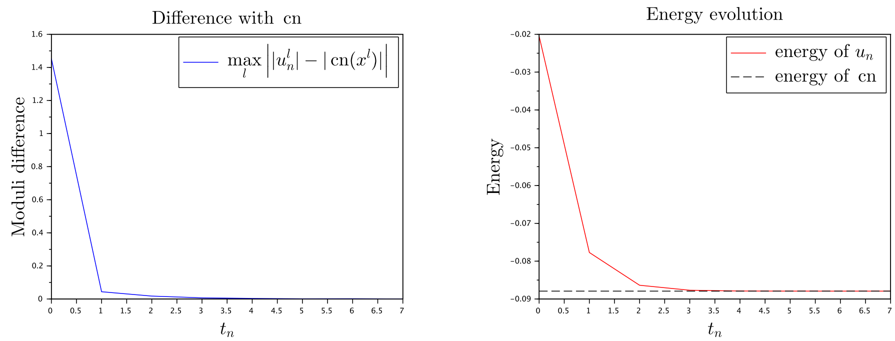

FIGURE 7.4. For $m = \mathcal{M}(\mathrm{cn}) = 2(E - (1-k^2)K)/k^2$, focusing, anti-periodic case

7.2.2. *The Defocusing Case.* We finally turn out to the defocusing case, still imposing anti-periodicity. We have used $b = -2k^2$ and $T = 4K$.

We have tested the algorithm without the momentum renormalization step (6.6) and confirmed the theoretical result Proposition 3.6, which states that a plane wave is the minimizer. We present the result in Figure 7.5 for initial data (c) of (7.1) and mass constraint $m = \mathcal{M}(\mathrm{sn}) = \frac{2(K-E)}{k^2}$. Note a plateau in the two graphs of Figure 7.5. This is due to the fact that the sequence remains for some time close to sn (which is the expected minimizer if we impose in addition the momentum constraint), before eventually converging to the plane wave minimizer.

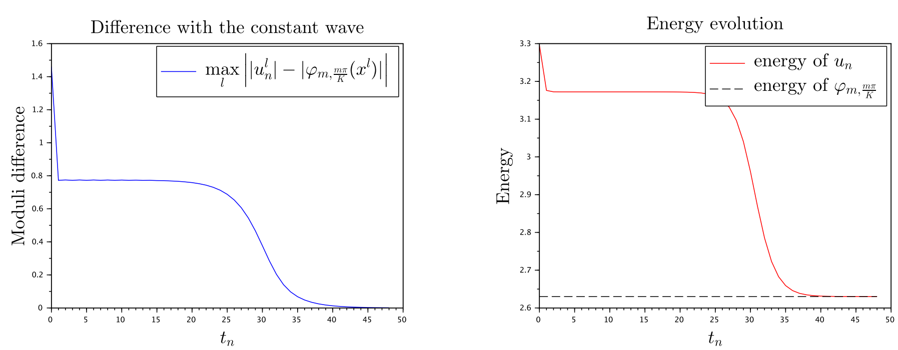

FIGURE 7.5. For $m = \mathcal{M}(\mathrm{sn}) = 2(E(k) - K)/k^2$, defocusing, anti-periodic case without momentum constraint

Finally, we run the full algorithm with mass and momentum renormalization for mass constraint $m = \mathcal{M}(\mathrm{sn}) = \frac{2(K-E)}{k^2}$ and 0 momentum constraint. No theoretical result is available in this case. We made the following observation, which confirms Conjecture 3.7.

Observation 7.1. *The function* sn *is a minimizer for problem* (3.5) *with* $m = \mathcal{M}(\mathrm{sn})$.

We present in Figure 7.6 the result of the experiment with full algorithm for initial data (c) of (7.1) and mass constraint $m = \mathcal{M}(\mathrm{sn}) = \frac{2(K-E)}{k^2}$.

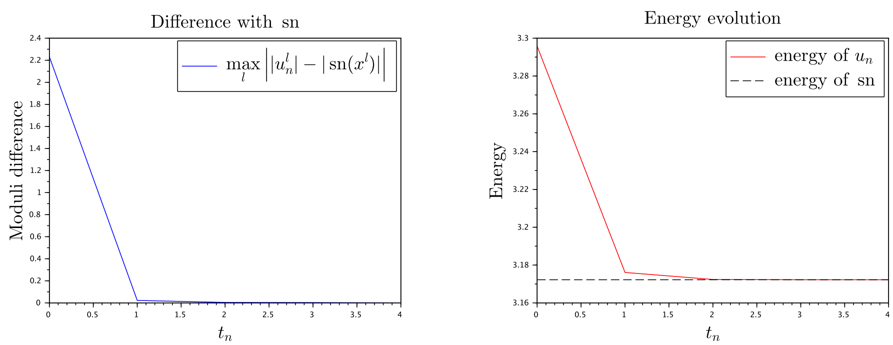

FIGURE 7.6. For $m = \mathcal{M}(\mathrm{sn}) = 2(E(k) - K)/k^2$, defocusing, anti-periodic case with momentum constraint

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